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STRANGE NONCHAOTIC ATTRACTOR IN LOW-FREQUENCY QUASIPERIODICALLY DRIVEN SYSTEMS

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To generate strange nonchaotic attractor in quasiperiodically driven systems, there must be an unstable region in its phase-space. In this paper, a theoretical analysis shows that the quasiperiodic force acts as noise to lead the trajectory running into different expanding orbits when the trajectory repeatedly runs into the unstable region. Thus the resulting attractor is strange. The local-phase Lyapunov exponent is introduced for the study of low-frequency quasiperiodically driven systems. It is shown that the local-phase Lyapunov exponents can be approximated by the exponents of autonomous systems. The statistical properties of SNA system driven by low-frequency quasiperiodic force can then be approached by a set of autonomous systems.

1. Introduction

Strange nonchaotic attractor (SNA) is a kind of attractor exhibiting some properties of periodic and chaotic systems [Grebogi *et al.*, 1984]. Like periodic attractors their typical trajectories exhibit no sensitive dependence on initial conditions; like usual chaotic attractors their geometric structures are complicated and so the trajectories can have a singular continuous power spectrum. SNA can be generated in nonautonomous ordinary differential equations driven by two incommensurate periodic forces or discrete maps driven by a quasiperiodic force. SNAs have also been observed in some physical systems [Ditto *et al.*, 1990; Zhou *et al.*, 1992; Ding *et al.*, 1997]. SNA can be quantitatively characterized by a variety of methods, including Lyapunov exponent, fractal dimension, rotation number and phase-sensitivity exponent [Romeiras *et al.*, 1987; Ding *et al.*, 1989; Pikovsky & Feudel, 1994]. The renormalization group has also been developed for the analysis of SNA [Kuznetsov *et al.*, 1998]. One of the important numerical observations is that the typical trajectories of SNA have positive finite-time Lyapunov exponents, although asymptotically, the time-independent Lyapunov exponent is negative [Pikovsky & Feudel, 1995]. Different mechanisms have been proposed for the creation of SNA in quasiperiodically forced systems [Kapitaniak, 1993; Heagy & Hammel, 1994; Yalcinkaya & Lai, 1996; Nishikava & Kaneko, 1996; Prasad *et al.*, 1997; Witt *et al.*, 1997]. Most of them involve a collision between stable and unstable regions. With these methods, SNAs typically appear in a narrow vicinity along the transition boundaries of chaos and torus. It has been shown that, if the driving frequency is sufficiently low, SNAs

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can occur in a large parameter region [Shuai & Wong, 1998, 1999].

It has been stressed that to generate an SNA with a quasiperiodic driving force there must be an unstable subregion in the phase space of the system [Heagy & Hammel, 1994; Lai, 1996; Pikovsky & Feudel, 1995]. However, if we replace the irrational frequency of the driving force with a nearby rational frequency, the unstable subregion still exists. But in this case SNA can no longer be observed. Thus an important question is what kind of role does the quasiperiodic force play in the creation of SNA. This question is also partly related to the argument in [Anishchenko *et al.*, 1996, 1997; Pikovsky & Feudel, 1997]. It was argued in [Shuai & Wong, 1998, 1999] that the quasiperiodic force acts as a noise in an SNA system. This conclusion was only based on some numerical simulations and limited to the case of low-frequency driving force. In Sec. 2 of this paper, a theoretical basis for this statement is considered generally. Theoretical analysis shows that, acting as a noise source, the quasiperiodic force drives the trajectory into different expanding orbits during different time intervals to create a strange attractor. SNAs in low-frequency quasiperiodically driven systems have been studied numerically [Shuai & Wong, 1998, 1999]. In Sec. 3, a new analysis of the local-phase Lyapunov exponent method is provided and gives us some new insights on such an SNA system. For a low-frequency driving system, its local-phase Lyapunov exponents can be approximated by the exponent of autonomous systems. Thus the statistical properties of such a system can be discussed with a set of autonomous systems. To confirm these discussions, a quasiperiodically driven logistic map is simulated in Sec. 4.

2. Finite-Time Lyapunov Exponent

In this paper the following different maps are investigated,

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \sin\theta_n) \tag{1a}$$

$$\theta_{n+1} = (\theta_n + 2\pi\omega) \mod 2\pi \tag{1b}$$

where $\mathbf{x}$ is a d-dimensional vector and $\mathbf{F}$ a d-dimensional vector function of $\mathbf{x}$. Suppose the attractor lives in the region of $(-1, 1)$ and the function $\sin\theta$ is of the first-order in $\mathbf{F}(\mathbf{x}, \sin\theta)$. There are $d+1$ Lyapunov exponents for Eq. (1). One of them, corresponding to Eq. (1b), is always zero. Assuming the other d Lyapunov exponents are all negative, there is no other periodic function in Eq. (1a). The d Lyapunov exponents can be obtained by the evolution of the tangent vector $\mathbf{y}_n$:

$$\mathbf{y}_{n+1} = \mathbf{DF}(\mathbf{x}_n) \cdot \mathbf{y}_n \tag{2}$$

where $\mathbf{DF}$ denotes the Jacobian matrix of partial derivatives of $\mathbf{F}$ to $\mathbf{x}$. The d Lyapunov exponents λ of the system are [Ott, 1993]

$$\lambda = \lim_{n\to\infty} \lambda_n(0) = \lim_{n\to\infty} \frac{1}{n} \ln|\mathbf{DF}^n(\mathbf{x}_0)\cdot\mathbf{y}_0| \tag{3}$$

with

$$\mathbf{DF}^n(\mathbf{x}_i) = \mathbf{DF}(\mathbf{x}_{n+i})\cdot\mathbf{DF}(\mathbf{x}_{n-1+i})\cdots \cdot\mathbf{DF}(\mathbf{x}_{i+1})\cdot\mathbf{DF}(\mathbf{x}_i) \tag{4}$$

Here, $\lambda_n(0)$ is called the time-n Lyapunov exponent from $t = 0$. In particular, one can define time-1 Lyapunov exponent $\lambda_1(n)$ at each time step n:

$$\lambda_1(n) = (n+1)\lambda_{n+1}(0) - n\lambda_n(0) \tag{5}$$

Then the time-τ Lyapunov exponent from time n_0 is

$$\lambda_\tau(n_0) = \frac{1}{\tau}\sum_{i=n_0}^{n_0+\tau-1} \lambda_1(i) \tag{6}$$

Suppose the initial point $(\mathbf{x}_0, \theta_0)$ is located in the attractor space. Furthermore, suppose at time m the trajectory is at point $(\mathbf{x}_m, \theta_m)$ with $\mathbf{x}_m - \mathbf{x}_0 < \delta\mathbf{x}$ and $\theta_m - \theta_0 < \delta\theta$. Here $\delta\mathbf{x}$ and $\delta\theta$ are at infinitesimal distance. Let the nearby points $(\mathbf{x}_m, \theta_m)$ be another initial point, i.e. $(\mathbf{x}'_0, \theta'_0) = (\mathbf{x}_m, \theta_m)$. Noticing that we have $\delta\theta_n \equiv \delta\theta_0$, in the following the evolution of the difference between $\mathbf{x}_0$ and $\mathbf{x}'_0$ is discussed. Applying Taylor series, one has

$$\delta\mathbf{x}_1 = \mathbf{DF}(\mathbf{x}_0)\cdot\delta\mathbf{x}_0 + \left.\frac{\partial\mathbf{F}}{\partial\sin\theta}\right|_{\theta_0} \cos\theta_0\delta\theta_0 + \mathbf{O}(\mathbf{x}_0, \theta_0) \tag{7}$$

Ignoring the second- and higher-order terms, the evolution of the difference is

$$\delta\mathbf{x}_n = \mathbf{DF}^n(\mathbf{x}_0)\cdot\delta\mathbf{x}_0 + \mathbf{A}\delta\theta_0 \tag{8}$$

where the vector

$$\mathbf{A} = \sum_{i=0}^{n-1} \mathbf{DF}^{n-1-i}(\mathbf{x}_{i+1})\cdot\left.\frac{\partial\mathbf{F}}{\partial\sin\theta}\right|_{\theta_i} \cos\theta_i \tag{9}$$

The negative nontrivial maximum Lyapunov exponent indicates $\mathbf{DF}^n(\mathbf{x}_0) = \mathbf{0}$ for large time n. If ω is rational, one can let $\delta\theta_0 = 0$. So we get $\delta\mathbf{x}_n = \mathbf{0}$, $\delta\theta_n = 0$. In this case a period-m attractor is observed. However, for the quasiperiodic function, we have $\theta_{n_1} \neq \theta_{n_2}$ for any $n_1 \neq n_2$ and thus $\delta\mathbf{x}_n = \mathbf{0}$, $\delta\theta_n = 0$ cannot be obtained.

For almost all of the numerically investigated SNA systems, the term $\sin\theta$ is of the first-order in the map. In this case,

$$\left.\frac{\partial\mathbf{F}(\mathbf{x}_n, \sin\theta)}{\partial\sin\theta}\right|_{\theta_n} = \mathbf{F}(\mathbf{x}_n, \cos\theta_n) = \mathbf{F}\left(\mathbf{x}_n, \sin\left(\theta_n + \frac{\pi}{2}\right)\right) \equiv \mathbf{x}''_{n+1} \tag{10}$$

Here $\mathbf{x}''_{n+1}$ is obtained from Eq. (1) with the initial conditions $\mathbf{x}''_0 = \mathbf{x}_0$ and $\theta''_0 = \theta_0 + \pi/2$.

The fact that d Lyapunov exponents are negative implies $\mathbf{DF}^n(\mathbf{x}_i) \to \mathbf{0}$ when $n \gg i$. Thus only the recent limited terms remain in the vector $\mathbf{A}$. Assuming that the remaining terms are from $n - \tau - 1$ to $n - 1$:

$$\begin{aligned}\mathbf{A} &= \mathbf{x}''_n \cos\theta_{n-1} + \mathbf{DF}(\mathbf{x}_{n-1}) \cdot \mathbf{x}''_{n-1}\cos\theta_{n-2} \\ &\quad + \cdots + \mathbf{DF}^\tau(\mathbf{x}_{n-\tau}) \cdot \mathbf{x}''_{n-\tau}\cos\theta_{n-\tau-1} \\ &= \sum_{i=0}^{\tau} \mathbf{DF}^i(\mathbf{x}_{n-i}) \cdot \mathbf{x}''_{n-i}\cos\theta_{n-i-1}\end{aligned} \tag{11}$$

and so

$$\delta\mathbf{x}_n = \delta\theta_0 \sum_{i=0}^{\tau} \mathbf{DF}^i(\mathbf{x}_{n-i}) \cdot \mathbf{x}''_{n-i}\cos\theta_{n-i-1}\,. \tag{12}$$

Then the maximum distance $|\delta\mathbf{x}_n|$ can be estimated as

$$|\delta\mathbf{x}_n| \leq \delta\theta_0 \sum_{i=0}^{\tau} |\mathbf{DF}^i(\mathbf{x}_{n-i}) \cdot \mathbf{x}''_{n-i}\cos\theta_{n-i-1}| \tag{13}$$

The time-i Lyapunov exponent is obtained from $\mathbf{DF}^i(\mathbf{x}_{n-i})$. Defining the maximum time-i Lyapunov exponent at time $n - i$ as $\lambda_i(n - i)$, and noticing $|\mathbf{x}''_n| \leq 1$,

$$|\mathbf{DF}^i(\mathbf{x}_{n-i}) \cdot \mathbf{x}''_{n-i}| \leq \exp(i \cdot \lambda_i(n - i))\,. \tag{14}$$

The equation holds only when the vector $\mathbf{x}''_{n-i}$ lies in the direction of the eigenvector of $\mathbf{DF}^i(\mathbf{x}_{n-i})$ which eigenvalue is $\lambda_i(n - i)$ and $|\mathbf{x}''_n| = 1$. The maximum distance $|\delta\mathbf{x}_n|$ is then mainly determined by the exponents $\lambda_1(n - 1)$, $\lambda_2(n - 2), \ldots, \lambda_i(n - i), \ldots$ and $\lambda_\tau(n - \tau)$, i.e.

$$|\delta\mathbf{x}_n| \leq \delta\theta_0 + \delta\theta_0 \sum_{i=1}^{\tau} \exp(i \cdot \lambda_i(n - i))\,. \tag{15}$$

According to Eq. (6),

$$\lambda_{i+1}(n - i - 1) = \frac{i}{i+1}\lambda_i(n - i) + \frac{1}{i+1}\lambda_1(n - i - 1)\,. \tag{16}$$

The value of $\lambda_{i+1}(n - i - 1)$ is mainly determined by the term $\lambda_i(n - i)$. It implies that the change of the exponents from $\lambda_i(n - i)$ to $\lambda_{i+1}(n - i - 1)$ is a little smooth. Considering the fact that the time-i Lyapunov exponent asymptotically approaches the negative Lyapunov exponent with the increase of time i, there are two typical situations for such changes. The first case is that all these finite-time Lyapunov exponents are negative. It means that the trajectory always runs in the contracting space. The terms $\exp(i \cdot \lambda_i(n - i))$ become smaller as i increases. So, the distance $|\delta\mathbf{x}_n|$ in Eq. (15) is of the order of $\delta\theta_0$. It means that the different points in the trajectory will converge to each other if their phase angles are close. The resultant attractor is a torus.

However, various studies have shown that there may exist an unstable region in the phase space for a nonchaotic system [Heagy & Hammel, 1994; Lai, 1996; Pikovsky & Feudel, 1995]. Therefore, the second case is that the trajectory runs into the unstable region with a high frequency during a time interval, e.g. (n_0, n_1). In this case, the finite-time Lyapunov exponents $\lambda_{n-n_0}(n_0)$ are positive and increase with the increase of n from n_0. After time n_1, the exponent reaches its maximum, i.e. $\lambda_0 = \lambda_{\tau_0}(n_0) = [\lambda_1(n_0) + \lambda_1(n_0 + 1) + \cdots + \lambda_1(n_1)]/\tau_0$ with $\tau_0 = n_1 - n_0 + 1$. Then, the trajectory runs back into the stable region with high frequency and $\lambda_{n-n_0}(n_0)$ gradually decreases to become negative. It should be noted that $\lambda_{n-n0}(n_0)$ discussed here is with the fixed beginning time n_0. In contrast, the exponent $\lambda_i(n - i)$ discussed in Eq. (15) is with a fixed ending time n. An increase in i means the backward time. However, despite this difference, a similar result can be drawn for $\lambda_i(n - i)$. As shown in Fig. 1, $\lambda_i(n_1 - i)$ increases with the increase of backward time i for the fixed ending time n_i. The maximum finite-time Lyapunov exponent

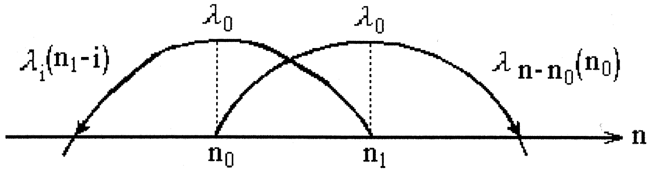

Fig. 1. Geometric diagram of the difference between the time-$(n-n_0)$ Lyapunov exponents $\lambda_{n-n_0(n_0)}$ with a fixed beginning time n_0 and the time-i Lyapunov exponents $\lambda_i(n-i)$ with a fixed ending time n. Their maximum finite-time Lyapunov exponents are λ_0 which is obtained in the time interval of (n_0, n_1).

is also reached in the interval of (n_1, n_0) and so its value is still λ_0. After that, $\lambda_i(n_1-i)$ gradually decreases. So, ignoring the folding dynamics, the distance in Eq. (15) can be of the order of

$$|\delta \mathbf{x}_n| \approx \delta\theta_0 \exp(\tau_0\lambda_0) . \tag{17}$$

If $\delta\theta_0 = 10^{-16}$ and $\lambda_0 = 0.01$, $|\delta \mathbf{x}_n| \approx 0.43$ or 0.06 be obtained with $\tau_0 = 3600$ or 3400 respectively. So, once the trajectory runs into the expanding region with a high frequency for a long time interval, the tiny difference provided by $\delta\theta$ can be enlarged exponentially. Thus, one of the notable characteristics of SNA systems is that the finite-time Lyapunov exponent can have a long-time positive tail [Pikovsky & Feudel, 1995].

Suppose that at time n_0 before running into the expanding region, the trajectory is at the point $(\mathbf{x}_{n0}, \theta_{n0})$. Because this point is located in the contracting region, the nonchaotic trajectory will approach it repeatedly with time and then runs into the expanding region. However, the quasiperiodic force always provides different values to disturb the trajectory. So the trajectory always starts from a different point before running into the expanding region. Furthermore, running in the expanding region, the trajectory is still disturbed by the different values of $\sin\theta_n$. Thus the quasiperiodic force acts as noise to create a strange attractor. The theoretical analysis shows that it is the general conclusion for any SNA system. If, instead of the irrational frequency, a nearby rational frequency is applied in the driving force, although the unstable region still occurs in the phase space, a strange attractor cannot be obtained because of the lack of noise-like signal. In short, an unstable region in phase space and a noise-like nonchaotic signal are two necessary elements for the creation of SNA. The reason why the SNA occurs in quasiperiodic driven systems is that the quasiperiodic function is the simplest nonchaotic function that can provide noise-like signal.

3. Low-Frequency Quasiperiodically Driven Systems

SNAs in low-frequency quasiperiodically driven systems are discussed numerically in [Shuai & Wong, 1998, 1999]. A new analysis is provided here for such systems. The force $\mathbf{F}(\mathbf{x}_n, \sin\theta_n)$ is periodic in θ with period 2π and the points of the attractor are uniformly distributed on the phase axis θ from 0 to 2π. One can divide such a 2π region into L subregions, i.e. $\Theta_l \equiv (2\pi(l-1)/L, 2\pi l/L)$ for $l = 1, 2, \ldots, L$. Then there are n/L points uniformly distributed in each subregion Θ_l when $n \to \infty$. To describe the contracting or expanding property in each subregion Θ_l, the local-phase Lyapunov exponent λ_{Θ_l} is defined as

$$\lambda_{\Theta_l} = \lim_{n\to\infty} \frac{L}{n} \sum_{\substack{i=1 \\ i\in\Theta_l}}^{n} \lambda_1(i) \tag{18}$$

During each driving period, there are about $1/(L\omega)$ points continuously falling into each subregion. For the case of low driving frequency, the value of $1/(L\omega)$ is large enough. Suppose in the subregion l we have $\lambda_{\Theta_l} > 0$. It means that when the driving force is in the subregion l, the trajectory is statistically driven to an unstable region. Any perturbation can be enlarged according to Eq. (17) with $\tau_0 = 1/(L\omega)$. The maximum nontrivial Lyapunov exponent is the average of these L local-phase Lyapunov exponents, i.e. $\lambda = 1/L \sum_{l=1}^{L} \lambda_{\Theta_l}$.

Now we show that the local-phase Lyapunov exponent λ_{Θ_l} can be approximated by the Lyapunov exponent $\Lambda(F_1)$ of the autonomous map $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_l)$ with the constant value $F_l = \sin(\pi(2l-1)/L)$. This result is followed by the following three approaches. (1) In the case of $L \gg 1$ the driving force $\sin(\theta)$ in each subregion Θ_l changes so small, i.e. from $\sin(2\pi(l-1)/L)$ to $\sin(2\pi l/L)$, that it can be treated as a constant driving force $F_l = \sin(\pi(2l-1)/L)$. (2) With $L \gg 1$, the difference is still small between of the values of F_l and F_{l+1}. So in most cases, i.e. if ignoring the crisis and bifurcation phenomena that can lead to the sudden change of the attractors, the geometric structures of the attractors of maps $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_l)$ and $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_{l+1})$ are quite similar. (3) Low enough frequency means $1/(L\omega) \gg 1$. After running about $1/(L\omega)$ points in subregion Θ_{l-1}, the next $1/(L\omega)$ points of Eq. (2) run into the subregion Θ_l. Thus, once the last $1/(L\omega)$ points are close to the attractor of $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_{l-1})$, the next

$1/(L\omega)$ points are naturally close to the attractor of $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_l)$. These approaches imply that, in most cases, the geometric structure of the trajectory in subregion Θ_l is similar to the attractor of the autonomous system $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_l)$. Having similar dynamic trajectory, the exponent λ_{Θ_l} can then be approximated by the exponent $\Lambda(F_l)$ of the map $\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, F_l)$.

As a result, the statistical properties of the low-frequency quasiperiodically driven system can be approximated by a set of autonomous systems $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, F_l)$ with $F_l = \sin(\pi(2l-1)/L)$ for $l = 1, 2, \ldots, L$. Suppose, among the L systems, the autonomous systems from l_1 to l_2 are chaotic, i.e. $\Lambda(F_{l_1}), \ldots, \Lambda(F_{l_2}) > 0$. Running into these subregions from Θ_{l_1} to Θ_{l_2}, the trajectory's dynamics are expanding. The local-phase Lyapunov exponent within these subregions is $\lambda_0 = (\Lambda(F_{l_1}) + \cdots + \Lambda(F_{l_2}))/(l_2 - l_1 + 1)$. During a driving period, the time interval for which the trajectory runs in these regions is $\tau_0 = (l_2 - l_1 + 1)/(L\omega)$. According to Eq. (17), any perturbation before running into the expanding region can be enlarged to the order of

$$\exp(\tau\lambda_0) = \exp\left(\sum_{l=l_1}^{l_2} \Lambda(F_l) \bigg/ L\omega\right) \tag{19}$$

In the case of $1/L\omega \gg 1$, this is a large number and a strange structure can be generated in these subregions.

This approach provides us with a simple method to distinguish SNA from torus in a nonchaotic system: If some $\Lambda(F_l)$ are positive, the nonchaotic attractor of system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \sin\theta_n)$ is strange. Because the selection of L is somewhat arbitrary, a simpler conclusion is that the SNA occurs for Eq. (1) if there is a chaotic phase for the system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, F)$ in the region $-1 \le F \le 1$. Driven by a force with frequency $\omega = (\sqrt{5}-1)/2$, the SNA typically appears in a narrow parameter region [Heagy & Hammel, 1994; Yalcinkaya & Lai, 1996; Prasad *et al.*, 1997; Witt *et al.*, 1997]. For a low driving frequency, if the system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \sin\theta_n)$ slowly oscillates in the chaotic and periodic boundaries of the system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, F)$ and if the periodic dynamics dominates over the chaotic dynamics, a nonchaotic but strange attractor can be observed. In this case, SNA can occur in a large parameter region. So an easy way to construct SNA in physical systems is to apply a low-frequency quasiperiodic force.

This approach also suggests that the Lyapunov exponent λ can be obtained from $\Lambda(F_l)$:

$$\lambda \approx \frac{1}{L}\sum_{l=1}^{L} \Lambda(F_l) \tag{20}$$

Driven by the lower frequency force, the dynamics of the system can still be approached by the same L autonomous systems. Thus the Lyapunov exponent λ is almost independent of the driving frequency if the frequency is small enough.

4. Simulation Results

A simulation example given as follows is discussed in this section to confirm the above discussions,

$$\begin{aligned} x_{n+1} &= \alpha(1 - \varepsilon\cos\theta_n)x_n(1-x_n) \\ \theta_{n+1} &= (\theta_n + 2\pi\omega) \mod 2\pi \end{aligned} \tag{21}$$

If the initial condition of θ_0 is replaced by $\theta_0 + \pi$, this system is the same as that discussed in [Heagy & Hammel, 1994; Prasad *et al.*, 1997]. Its time-1 Lyapunov exponent is

$$\lambda_1(n) = \ln[a(1 - \varepsilon\cos\theta_n)] + \ln(|1 - 2x_n|) \tag{22}$$

It has been shown that there is a small region in the $a - \varepsilon$ parameter plane with a nonzero measure where the SNA exists between the torus and chaos when $\omega = (\sqrt{5}-1)/2$ [Prasad *et al.*, 1997]. The case for low-frequency driving force is discussed here. Simulation results show that a nonchaotic attractor can be obtained with nontrivial Lyapunov exponent $\lambda = -0.2789$ for $\alpha = 3.25$, $\varepsilon = 0.1$ and $\omega = 10^{-6}\sqrt{2}$. The trajectory is shown in Fig. 2(a). In Fig. 2(b), its corresponding finite-time Lyapunov exponents $\lambda_\tau(n)$ are given with the fixed time interval $\tau = 50$. The values of $\lambda_\tau(n)$ repeatedly become positive with time, indicating that the trajectory runs into the expanding region repeatedly.

In order to discuss the diverging dynamics in detail, the finite-time Lyapunov exponent $\lambda_\tau(n_0)$ versus τ from $n_0 = 331,500$ is given in Fig. 3(a). A long positive finite-time Lyapunov exponent can be observed clearly in this figure. Around $n = 332,000$, $\theta_n = 169°$ (as shown with arrow **A** in Fig. 3), the trajectory runs into the expanding region. When $n = 373,000$ (i.e. $\tau = \tau_0 = 41,000$), the finite-time Lyapunov exponent approaches its maximum $\lambda_0 \approx 0.06$. It means that any perturbation with the order of $\exp(-\tau_0\lambda_0)$ provided by x_n or

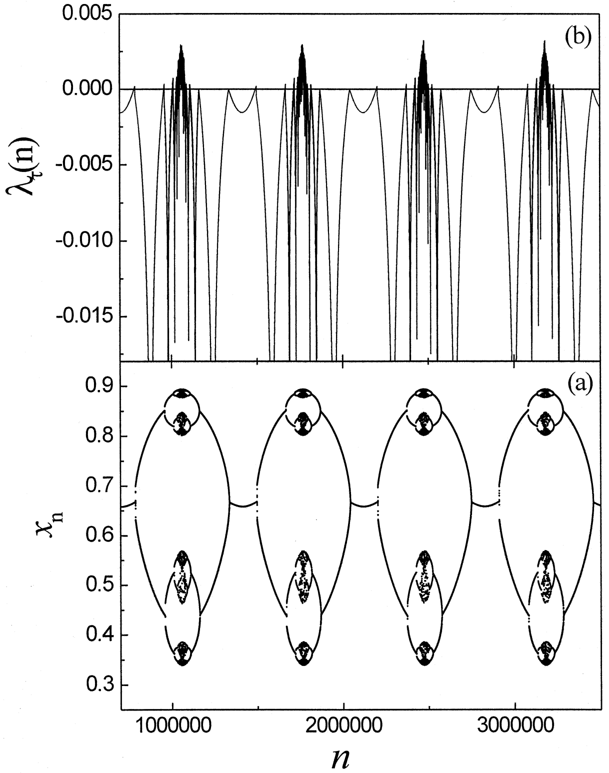

Fig. 2. Simulation result of Eq. (21) for $\alpha = 3.25$, $\varepsilon = 0.1$ and $\omega = 10^{-6}\sqrt{2}$. (a) The trajectory from time 702,000 to 3,500,000. (b) Its corresponding time-τ Lyapunov exponents with $\tau = 50$. The initial 702,000 points are ignored.

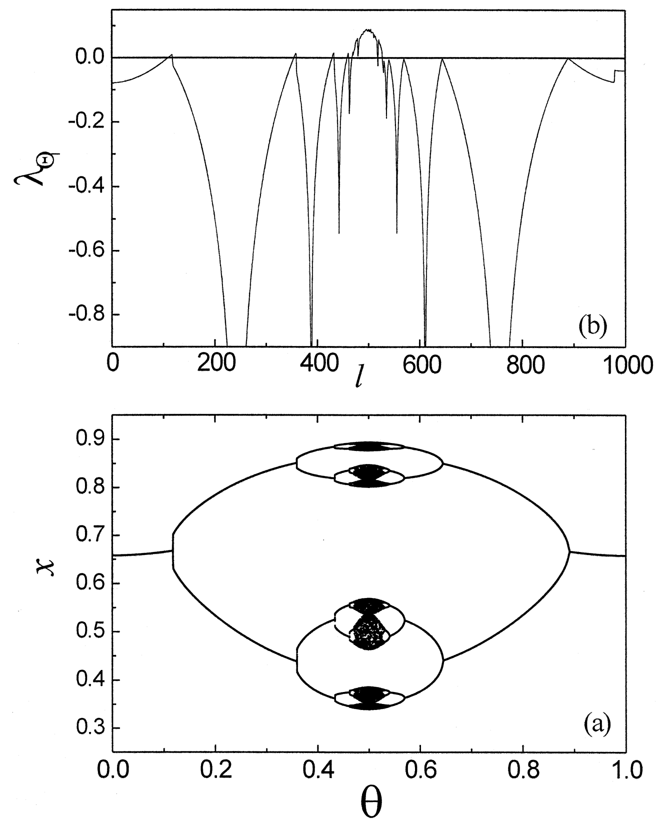

Fig. 4. (a) Attractor in the $x - \theta$ plane. (b) Its local-phase Lyapunov exponents λ_{Θ_l} for $l = 1, 2, \ldots, L$ with $L = 1000$.

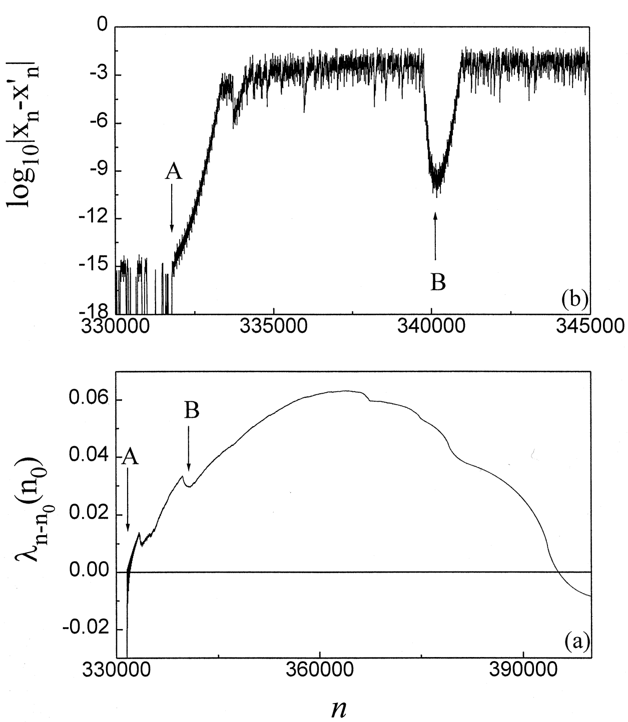

Fig. 3. (a) Finite-time Lyapunov exponent $\lambda_\tau(n_0)$ versus τ from $n_0 = 331,500$. (b) The difference between two points x'_n and x_n versus time n from 330,000 to 345,000. Around the point **A**, the trajectory runs into the expanding region.

θ_n at time $n = 332,000$ can be enlarged to the order of 1 at time $n = 373,000$. At time $n = 330,000$, the trajectory is at the point with $x_n = 0.34383\ldots$, $\theta_n = 0.46669\ldots$. Suppose there is a nearby point with $x'_n = x_n$ and $\theta'_n = \theta_n + 10^{-16}$. The difference between x'_n and x_n with time n is given in Fig. 3(b). One can see that the differences are of the order of 10^{-16} when the trajectories are in the contracting region. After $n = 332,000$, the trajectories are driven to expanding region and their differences are exponentially enlarged to reach the order of 0.01.

The attractor structure in $x - \theta$ plane is shown in Fig. 4(a). Driven by the low-frequency force, the dynamics of the trajectory are quite similar during different periods. In particular, the dynamics are expanding when the driving angles θ_n are in the region of $0.47 < \theta < 0.53$. The dynamics can be described by the local-phase Lyapunov exponents, as shown in Fig. 4(b) with $L = 1000$. In the region of $\lambda_{\Theta_l} > 0$, the strange structure occurs. If, instead of the irrational frequency, let $\omega = 1.414 \times 10^{-6}$, the periodically positive finite-time Lyapunov exponents can still be observed. But because there is no noise-like signal, an exactly periodic trajectory is obtained.

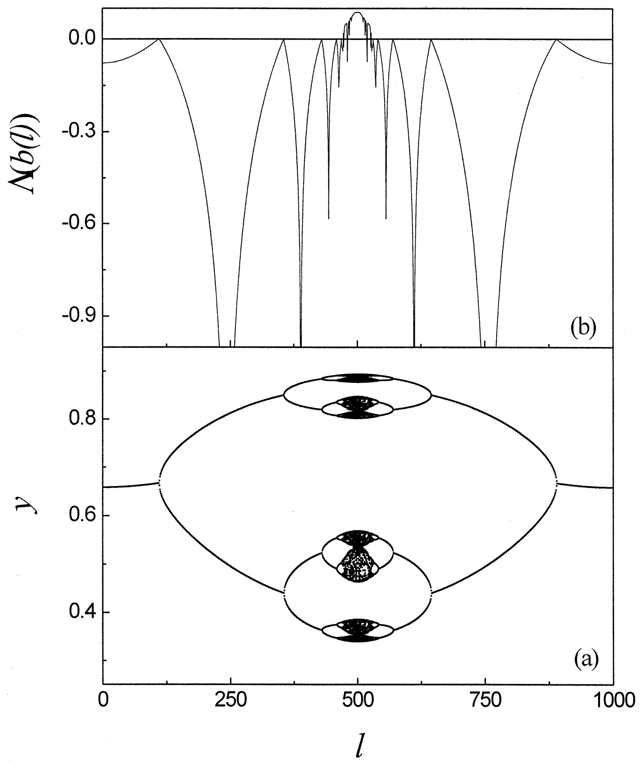

Fig. 5. (a) Bifurcation diagram and (b) the Lyapunov exponents of the logistic map $y_{n+1} = by_n(1 - y_n)$ for $b = \alpha[1 - \varepsilon\cos(\pi(2l-1)/L)]$ with $l = 1, 2, \ldots, L$. Here $L = 1000$.

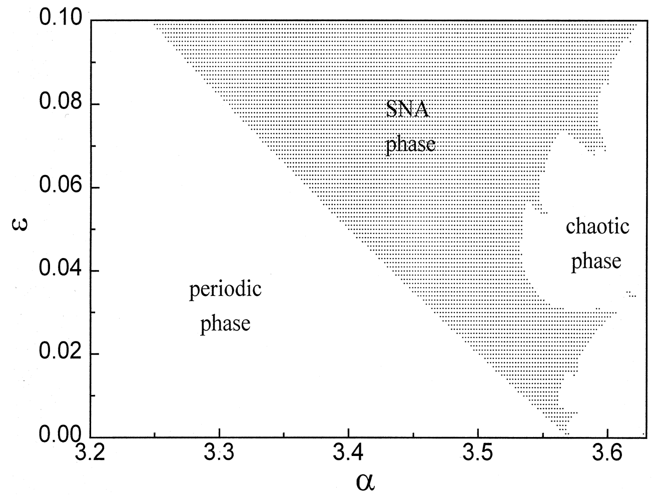

Fig. 6. Phase diagram of Eq. (21) in the $\alpha - \varepsilon$ plane. The dotted region is SNA phase.

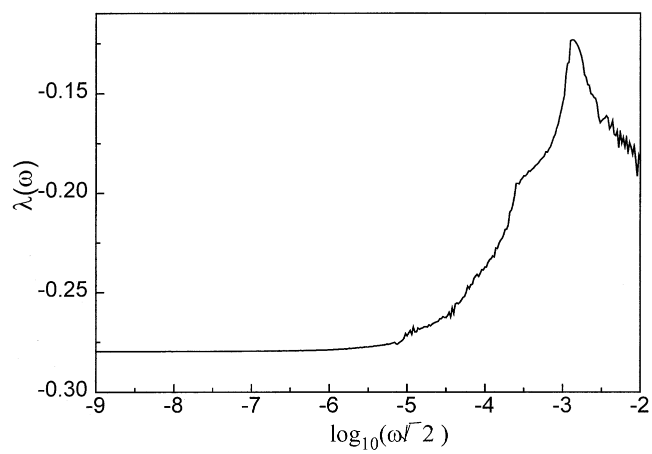

Fig. 7. Lyapunov exponent λ of Eq. (21) versus driving frequency from $10^{-9}\sqrt{2}$ to $10^{-2}\sqrt{2}$.

With a low driving frequency, the parameter $\alpha(1 - \varepsilon\cos\theta_n)$ gradually increases from $\alpha(1 - \varepsilon)$ to $\alpha(1 + \varepsilon)$ and then decreases to $\alpha(1 - \varepsilon)$ during a driving period. Thus, the attractor structure in $x - \theta$ plane can be approached by the bifurcation diagram of the logistic map $y_{n+1} = b_l y_n(1 - y_n)$ for $b_l = \alpha[1 - \varepsilon\cos(\pi(2l - 1)/L)]$ with $l = 1, 2, \ldots, L$, as shown in Fig. 5(a). The corresponding Lyapunov exponent versus l is given in Fig. 5(b). As expected, Figs. 5(a) and 5(b) are quite similar to Figs. 4(a) and 4(b) respectively. According to Eq. (20), the Lyapunov exponent of the system (21) is about -0.2794 which is within 0.2% of the exact value -0.2789. The chaotic region occurs when $b > b_0 = 3.5699\ldots$ for the logistic map. Thus, the expanding region appears whenever $\alpha(1 + \varepsilon\cos\theta_n) > b_0$. With $\alpha = 3.25$ and $\varepsilon = 0.1$, the expanding region is $0.47 < \theta < 0.53$.

For Eq. (21) the chaotic and nonchaotic phases in $\alpha - \varepsilon$ plane can be distinguished by checking the sign of the Lyapunov exponent λ. To distinguish SNA from torus, the following method is used: If $\alpha(1 + \varepsilon) > b_0$, some approaching logistic maps are chaotic and so the attractor must have a strange structure. Accordingly, Fig. 6 gives the phase diagram of Eq. (21) in $\alpha - \varepsilon$ plane. Different from the results in [Prasad *et al.*, 1997], SNA occurs here in a large region.

The Lyapunov exponent λ of Eq. (21) versus the driving frequency is shown in Fig. 7. As expected, the Lyapunov exponents are almost constant with the low frequency. They only change within 2.8% when the driving frequency decreases from 10^{-5} to 10^{-9}. For the logistic map, the periodic windows are dense throughout the chaotic range [Ott, 1993]. That is, given a value of b for which the orbit is chaotic, then in any ε neighborhood of that b value $[r - \varepsilon, r + \varepsilon]$, one can always find periodic windows no matter how small ε is. For example, as shown in Fig. 3, when the time is around 340,000, $\alpha(1 + \varepsilon\cos\theta_n)$ falls into a periodic window. Thus the difference is contracted to the order of 10^{-10} as shown with arrow **B** in Fig. 3(b). With the lower frequency in Eq. (21), the smaller periodic windows of the logistic map can be detected and more fluctuations of the local-phase Lyapunov exponents appear. However, because the Lyapunov exponent of Eq. (21) is an average of a set of logistic maps, this kind of oscillation is averaged and an almost constant Lyapunov exponent is obtained.

5. Conclusion

In this paper, the dynamics of SNA are discussed theoretically for quasiperiodically driven systems. An unstable region in phase space and a noise-like nonchaotic signal are two necessary elements for the creation of SNA. The reason why the SNA typically appears in quasiperiodic driven systems is that the quasiperiodic function is the simplest nonchaotic noise-like signal. For the low-frequency quasiperiodically driven system, the local-phase Lyapunov exponent is defined to discuss its properties. Thus, a set of autonomous systems can be applied to approach such a system. If some of the autonomous systems are chaotic, the nonchaotic nonautonomous system must be SNA.

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