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# Truncated chaotic trajectories in periodically driven systems with largely converging dynamics

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**Abstract**

Dynamical properties of numerically truncated trajectories are discussed for periodically driven chaotic systems with largely converging dynamics. Various trajectories having different initial conditions can be contracted so close to each other that they are truncated into a single pseudotrajectory. It is more easily observed numerically if the driving frequency is lower. Driven by an exactly periodic force, a periodic pseudotrajectory is obtained. Its period can be quite short and independent of truncation error. 

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## 1. Introduction

Computers are indispensable in the study of chaotic systems. However, because of the exponential sensitivity of chaotic solutions to noise, small errors in the solution can grow very rapidly with time such that the noise due to truncation of computers can drastically change the solution (pseudotrajectory) from the true trajectory of chaotic systems. For pseudotrajectories, an important issue is the shadowability, which deals with the existence of true trajectories that stay near pseudotrajectories [1–6]. It has been shown that for hyperbolic or nearly hyperbolic chaotic systems there exists a true trajectory with a slightly different initial condition that shadows the pseudotrajectory for a long time. For nonhyperbolic chaotic systems in which the finite-time Lyapunov exponent fluctuates about zero, the greater the finite-time fluctuation about zero, the smaller the power law exponent, resulting in large shadowing distances and valid trajectories of limited length [7,8].

Truncation not only induces noise that disturbs chaotic trajectories, but also divides the continuous phase space into a discrete lattice so that chaotic orbits must collapse to periodic cycles if the time is

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sufficiently long [9–12]. The latter effect of truncation is called the collapsing effect [13]. Statistical properties of truncated periodic cycles, e.g., the expected period length and expected number of periodic orbits, have been investigated for various chaotic systems [9–13]. Some phenomenological models, e.g. random mappings with a single attracting center, have been proposed to provide a theoretical analysis on these statistical properties [13–15].

Discussions of truncation effects on chaotic systems are currently concentrated on either hyperbolic or nonhyperbolic autonomous systems. Numerical simulations with these autonomous systems show that the mean period of truncated periodic cycles is a power function versus truncation error [11]. For these systems the finite-time Lyapunov exponents have minimal fluctuation about the Lyapunov exponent. In this Letter, we discuss dynamical properties of truncated trajectories for periodically driven chaotic systems, in which the largest nontrivial finite-time Lyapunov exponent fluctuates about zero to a greater extent. Such chaotic systems repeatedly show strong converging dynamics. Compared to the autonomous systems discussed in Refs. [9–15], pseudotrajectories of such nonautonomous systems possess some different properties. Various trajectories will converge and be truncated into a single pseudotrajectory (SPT). If the driving force is exactly periodic, a rather short periodic cycle can be obtained, whose period can be independent of truncation error in some scales.

## 2. Single pseudotrajectory

SPTs can typically be observed in systems driven by a periodic force to oscillate between chaotic and periodic states. Consider an autonomous system

$$\boldsymbol{x}(t+1)=\boldsymbol{F}(\boldsymbol{x}(t),\beta) \tag{1}$$

with control parameter $\beta$. Suppose chaotic attractors are obtained for $\beta$ in the region $(\alpha_0,\alpha_1)$ and periodic attractors are obtained for $\beta\in(\alpha_1,\alpha_2)$. Now let $\beta$ be variable and driven by a periodic force:

$$\boldsymbol{x}(t+1)=\boldsymbol{F}(\boldsymbol{x}(t),\beta(t))$$

$$\beta(t)=\beta_0+\beta_1\sin(2\pi ft) \tag{2}$$

with $\beta_0,\beta_1>0$. If $\beta_0-\beta_1\in(\alpha_0,\alpha_1)$ and $\beta_0+\beta_1\in(\alpha_1,\alpha_2)$, then the dynamics of this system alternate between expanding and contracting behaviors. When, on average, the expanding dynamics are stronger than the contracting dynamics, the resultant attractor is chaotic. With a sufficiently low frequency, the time interval that $\beta$ remains in the region $(\alpha_1,\alpha_2)$ during each driving period is also long enough. Under long continually converging dynamics, various trajectories will be converged and at last truncated to a single trajectory once their differences are smaller than the truncation error. As a result, an SPT is obtained.

As an example we discuss a periodically driven logistic map:

$$x(t+1)=ax(t)(1-x(t))+A\sin(2\pi ft) \tag{3}$$

with $a=3.6$ and $f=0.005$. The system has two Lyapunov exponents, one of which is always zero, corresponding to the periodic driving force. Simulation results show that chaotic attractors can be obtained when $A$ is slightly smaller than 0.1178. A plot of the nontrivial Lyapunov exponent $\lambda$ versus amplitude $A$ is shown in Fig. 1. In region III ($0.1005<A<0.115$), SPTs are always observed for truncation error of $10^{-18}$. In region I, SPTs are seldom observed while region II is a transition region. To detect an SPT, various trajectories with different initial conditions $x'(0)$ are generated randomly and the difference $\Delta x(t)=x(t)-x'(t)$ was calculated for $x(0)=0.2$. An SPT is defined by the condition that all $\Delta x(T)$ numerically go to zero before time $T=10^6$.

For example, with $A=0.11$ the SPT can be obtained before $t=3800$. Its dynamics are shown in Fig. 2. Fig. 2(a) shows the driving force. In Fig. 2(b), the trajectory for $x(0)=0.2$ is drawn. Fig. 2(c) gives the time-$\tau$ Lyapunov exponents for $\tau=5$. The time-$\tau$ Lyapunov exponent $\lambda^\tau(t)$ is defined as

$$\lambda^\tau(t_0)=\frac{1}{\tau}\sum_{t=t_0}^{t_0+\tau-1}\ln\left|\frac{\mathrm{d}f(x)}{\mathrm{d}x(t)}\right|=\frac{1}{\tau}\sum_{t=t_0}^{t_0+\tau-1}\lambda^1(t). \tag{4}$$

When the force $A\sin(2\pi ft)$ is positive, the finite-time Lyapunov exponents are typically positive and the

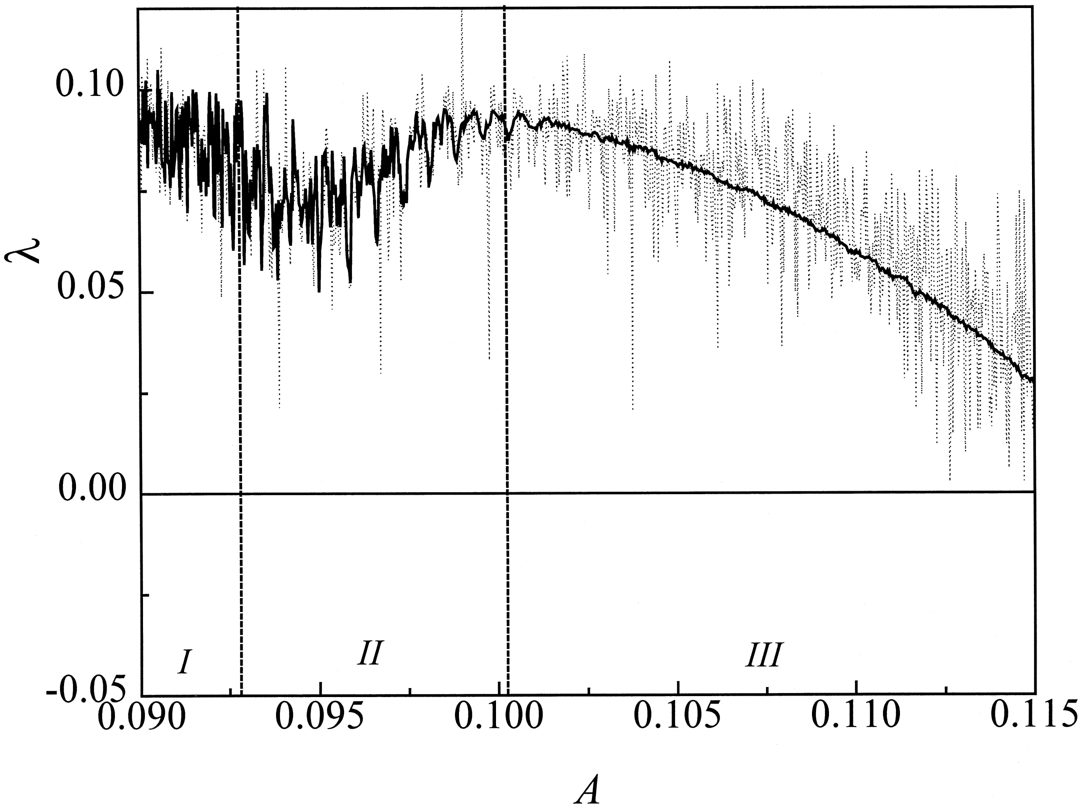


Fig. 1. Lyapunov exponent $\lambda$ versus amplitude $A$ from 0.09 to 0.115. The solid line of Lyapunov exponent is calculated by directly applying $\sin(2\pi ft)$ as a driving force. The dotted grey line is obtained by applying an exactly periodic force. SPT is seldom observed in region I and always observed in region III; region II is a transition region.

trajectory expands to a large region. When the force is negative, the exponents become typically negative and a period-2 structure is obtained. The sign of $\lambda^{\tau}(t)$ reflects the diverging or contracting behavior for time from $t$ to $t+\tau$. Although the value of $\lambda^{\tau}(t)$ varies with the choice of $\tau$, the time-5 Lyapunov exponent adequately characterizes the dynamics of the present system as shown in Fig. 2. In fact, any $\tau$ with $\tau \ll 1/f$ can be used to discuss the detailed properties of trajectories within one driving period. This is because when the trajectory is in the diverging (contracting) region, the time-1 Lyapunov exponents $\lambda^1$ and so $\lambda^{\tau}(t)$ become typically positive (negative). Fig. 2(d) shows the difference $\Delta x(t)$ with time between two trajectories with initial conditions of $x(0)=0.2$ and $x'(0)=0.7$, respectively. It can be seen clearly that the difference often decreases when $\lambda^{\tau}(t)$ is negative and starts increasing when $\lambda^{\tau}(t)$ becomes positive. In the contracting period, two trajectories have a chance to approach each other. They are truncated to the same trajectory at $t=1170$.

Now we show that exponentially higher precision is required in computation to avoid SPT when the driving frequency becomes lower. Suppose the typical time interval of the contracting movement is $T_n$. The dynamics during the interval $T_n$ is associated with time-$T_n$ Lyapunov exponent $\lambda^{T_n}$. During this interval a vector with size on the order of the phase space, i.e. 1, can be compressed to the order of $\exp(\lambda^{T_n}T_n)$. Define the critical size $l_c = \exp(\langle \lambda^{T_n}T_n \rangle)$, where $\langle\rangle$ means the statistical average for various contracting time intervals. If the truncating error of computer is larger than $l_c$, various trajectories can be converged into an SPT. In Fig. 3, simulation results

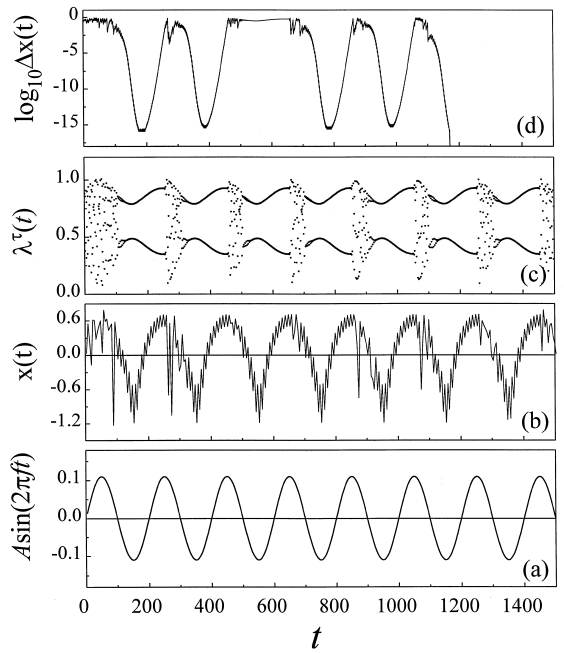


Fig. 2. Dynamics of SPT: (a) the driving force with $A = 0.11$, (b) the trajectory starting from $x(0) = 0.2$, (c) time-$\tau$ Lyapunov exponent with $\tau = 5$, and (d) the absolute difference between two trajectories starting from $x(0) = 0.2$ and $x'(0) = 0.7$. Around $t = 1170$, $\Delta x(t)$ numerically becomes zero.

about $\langle \lambda^{T_n} T_n \rangle$ versus $f$ are plotted in log-log axes. A linear fit indicates that

$$-\langle \lambda^{T_n} T_n \rangle \propto f^k \tag{5}$$

with $k = -1(\pm 0.002)$. So there is an exponential law

$$l_c \propto \exp(-1/f). \tag{6}$$

As a result, at lower driving frequency, an exponentially higher precision is required in computation to avoid SPT. With a fixed truncation error, SPT can be more easily observed numerically for a lower-frequency periodically driven system.

In fact, the exponential relation of Eq. (6) is general for the system given in Eq. (2). The low-frequency means that the driving force almost keeps constant within a short time interval. Thus the time-$\tau$ Lyapunov exponent $\lambda^{\tau}(t_0)$ at time $t = t_0$ with $\tau \ll 1/f$ can be approximated by the Lyapunov exponent of the autonomous system (1) driven by a constant force $\beta_{\mathrm{const}} = \beta_0 + \beta_1 \sin(2\pi f t_0)$. Notice that the Lyapunov exponent of the autonomous system (1) is typically negative when $\beta_{\mathrm{const}}$ is in the periodic region $(\alpha_1, \alpha_2)$. This indicates that the finite-time Lyapunov exponents $\lambda^{\tau}(t)$ of system (2) typically become negative when $\beta_0 + \beta_1 \sin(2\pi f t) \in (\alpha_1, \alpha_2)$. Thus, the time interval $T_n$ is the time interval that the driving force spends in the region of $(\alpha_1, \alpha_2)$. Suppose there are $k$ finite-time Lyapunov exponents $\lambda^{\tau}(t)$ in this time interval, i.e. $T_n \approx k\tau$. According to Eq. (4), the time-$T_n$ Lyapunov exponent $\lambda^{T_n}$ can be given as

$$\lambda^{T_n} = \frac{\tau}{T_n} \sum_{i=0}^{k-1} \lambda^{\tau}(t_1 + i\tau) \tag{7}$$

where $t_1$ is the beginning time at which the force $\beta_0 + \beta_1 \sin(2\pi f t)$ runs into the region $(\alpha_1, \alpha_2)$. Now, for a lower frequency $f' = \rho f$ with $\rho < 1$, the time interval is $T_n' = T_n/\rho$. Similarly, $\lambda^{T_n'}$ can also be given by $k$ finite-time Lyapunov exponents $\lambda^{\tau'}(t)$

$$\lambda^{T_n'} = \frac{\tau'}{T_n'} \sum_{i=0}^{k-1} \lambda^{\tau'}(t_1' + i\tau') \tag{8}$$

with $\tau' = \tau/\rho$. Here $t_1'$ is the time at which the force $\beta_0 + \beta_1 \sin(2\pi f' t)$ runs into $(\alpha_1, \alpha_2)$. Obviously, the finite-time Lyapunov exponent $\lambda^{\tau}(t_1 + i\tau)$ is equal to $\lambda^{\tau'}(t_1' + i\tau')$, since both of them can be approxi-

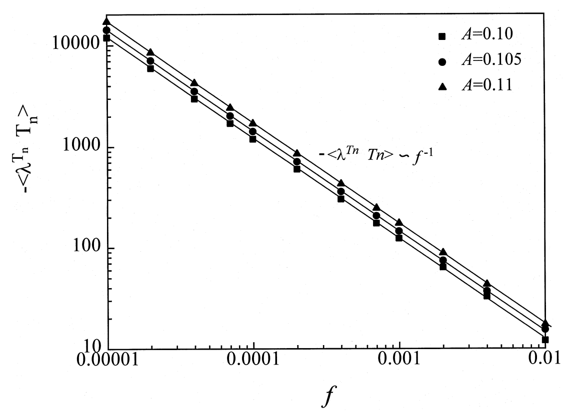


Fig. 3. Logarithm of $-\langle \lambda^{T_n} T_n \rangle$ versus logarithm of frequency $f$ of sinusoidal force, for several values of the amplitude $A$. Here, the slope of the fitted line is $-1$ ($\pm 0.02$).

mated by the autonomous map driven by the same constant force $\beta_{const} = \beta_0 + \beta_1 \sin(2\pi f(t_1 + i\tau)) = \beta_0 + \beta_1 \sin(2\pi f'(t_1' + i\tau'))$. As a result, $f\lambda^{T_n} T_n \approx f'\lambda^{T_n'} T_n'$. Considering the negative sign of $\lambda^{T_n}$, Eq. (6) follows.

## 3. Numerically periodic SPT

The statistical properties of truncated periodic trajectories have been discussed in detail for some one- or two-dimensional chaotic maps in Ref. [11]. For example, the mean period of the truncated cycle with truncation error $\epsilon$ is as $\epsilon^{-1/2}$ for the logistic map $x_{n+1} = 1 - 2x_n^2$ [11]. So if $\epsilon = 10^{-18}$, the mean period is as long as $10^9$. For a truncated cycle, if its period is sufficiently long, the resulting density distribution of pseudotrajectory can approach the real invariant density. As a result the properties of chaotic system can still be discussed with such a pseudotrajectory. In this section we show that for a chaotic system driven by an exactly periodic force with low frequency, the period of the periodic SPT may be short, on the order of $10^3$, and may be independent of truncation error in some scales.

Suppose the driving frequency $\omega$ is rational. Then an integer period $T_0 = n/\omega$ can be obtained for a suitably selected integer $n$. For system (2), suppose within time $T$ the SPT is obtained for various trajectories with different initial conditions. Then for function

$$\boldsymbol{x}(T + t_0) = \boldsymbol{F}^T(\boldsymbol{x}(t_0), \beta(t_0)) = \underbrace{\boldsymbol{F} \cdot \ldots \cdot \boldsymbol{F}}_{T}(\boldsymbol{x}(t_0), \beta(t_0)) \qquad (9)$$

we have

$$\frac{\partial \boldsymbol{x}(T + t_0)}{\partial \boldsymbol{x}(t_0)} = 0 \qquad (10)$$

for various $\boldsymbol{x}(t_0)$. So

$$\boldsymbol{x}(T + t_0) = \boldsymbol{G}(\beta(t_0), T) \qquad (11)$$

For $t_0 = 0$ and $t_0 = T_0 T$, we have

$$\boldsymbol{x}(T) = \boldsymbol{G}(\beta(0), T)$$

$$\boldsymbol{x}(T + T_0 T) = \boldsymbol{G}(\beta(T_0 T), T) \qquad (12)$$

respectively. Noticing $\beta(t) = \beta(t + T_0)$, the following equation is obtained,

$$\boldsymbol{x}(T + T_0 T) = \boldsymbol{x}(T). \qquad (13)$$

So an SPT is a periodic trajectory with a period of at least $T_0 T$. The exponential law in Eq. (6) indicates that, with a sufficiently low frequency, the various trajectories for system (2) will experience a long time of purely converging dynamics during a driving period. In this case, the trajectory may be contracted to the same truncated orbit in each driving period and so the period of SPT is equal to the driving period.

First, consider the logistic map driven by a square function. For the logistic map $x(t+1) = ax(t)(1 - x(t))$, a fixed point $x_F = 0.65517\cdots$ is obtained with Lyapunov exponent $\lambda_1 = -0.105$ for $a = a_1 = 2.9$. Simulation results show that various trajectories converge to the fixed point $x_F$ within 350 iterations. On the other hand, a chaotic attractor with $\lambda_2 = 0.183$ is obtained for $a = a_2 = 3.6$. Within 320 iterations, any difference on the order of $10^{-17}$ can approach a value of 0.1. For these two attractors, the transient time of a trajectory from any initial point to the attractor basin should be no more than 400. Now, a periodic square wave is applied to control the variable $a$. In particular, $a$ periodically alternates between $a_1$ and $a_2$ with time intervals $T_1$ and $T_2$, respectively. If $T_1$ and $T_2$ are much larger than the transient time, the finite-time Lyapunov exponents in time intervals $T_1$ and $T_2$ can be approximated by $\lambda_1$ and $\lambda_2$, respectively. The nontrivial Lyapunov exponent $\lambda$ of the system can be estimated by [16]

$$\lambda = \frac{T_1 \lambda_1 + T_2 \lambda_2}{T_1 + T_2}. \qquad (14)$$

Let $T_1 = T_2 = 2000$. Eq. (14) suggests that the nontrivial Lyapunov exponent should be 0.039, indicating a chaotic attractor is generated. However, obviously, numerical trajectories always converge to $x_F$ during each $T_1$ time interval. So the period of this periodic SPT is 4000. Furthermore, the period is independent of truncation error changing from the order of $10^{-18}$ to $10^{-9}$. Following this periodic SPT, simulation results show that $\lambda = 0.041(\pm 0.001)$.

The short periodic SPT gives an incorrect density distribution. Following such an SPT, one cannot correctly discuss the properties of the chaotic system, including the Lyapunov exponent. However, it has been suggested that the collapsing effect can be prevented by the addition of noise [12,15]. In the present case, with the disturbance of micro-noise upon the periodic SPT, an ergodic trajectory is obtained. Driven by the same periodic force and same noisy time series, this ergodic trajectory is still an SPT. However, a reasonable supposition is that the real invariant density of the system can be approached by such a ergodic trajectory. Thus, the statistical properties, e.g, the Lyapunov exponent, of the real chaotic trajectory can be approximately reproduced with it. To approach chaotic ergodicity, random noise on the order of $10^{-18}$ is always added to the variable $x(t)$. With this process, the numerical result of the Lyapunov exponent is $\lambda = 0.039(\pm 0.001)$.

Now we discuss the Lyapunov exponent of the example discussed in Section 2. With $A = 0.11$, the FPT obtained is periodic with period 2600. To see this clearly, the SPT trajectory is given in Fig. 4(a) in the $x$–$\theta$ plane with

$$\theta(t+1) = \theta(t) + f \bmod 1. \tag{15}$$

In the Figure 20 000 points are drawn from $t = 10\,000$ to 30 000 with $x(0) = 0.2$. However, rather than a chaotic attractor, a periodic attractor is obtained numerically. Simulation also shows that the period is independent of truncation error changing from the order of $10^{-18}$ to $10^{-9}$. Following such a periodic SPT, the nontrivial Lyapunov exponent is $\lambda = 0.053(\pm 0.001)$. Following application of noise on the order of $10^{-18}$ to the system, the resulting trajectory has a Lyapunov exponent of $0.061 \pm 0.002$. In comparison, the value of 0.053 has an error more than 13%. In fact, the simulation results given in Fig. 1 are all calculated with the disturbance of noise in order to obtain an ergodic trajectory. The Lyapunov exponent calculated with the periodic SPT is also presented in Fig. 1 with a dotted grey line. One can see that the difference of Lyapunov exponents ob-

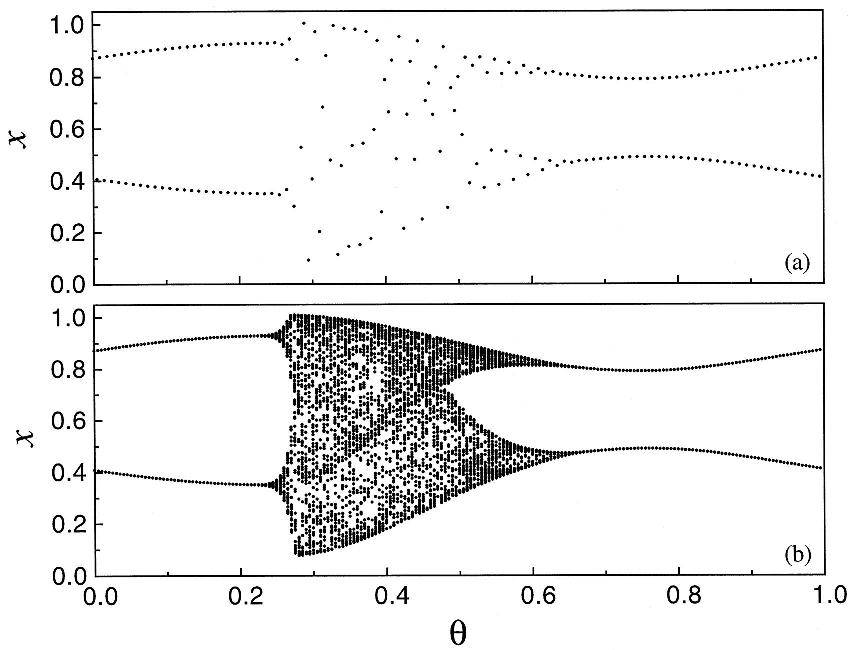


Fig. 4. SPT in the $x$–$\theta$ plane for Eq. (3) with $a = 3.6$, $A = 0.11$ and $f = 0.005$. (a) An exactly periodic force is applied. (b) A disturbed periodic force is applied. In the Figure, 20 000 points are drawn from $t = 10\,000$ to 30 000 with $x(0) = 0.2$.

tained from the two kinds of trajectories is quite obvious in the region of SPT, region III. The consistency of the two Lyapunov exponents in region I suggests that the real Lyapunov exponent of Eq. (3) can be approximately calculated with the distorted periodic force.

Because the finite-time Lyapunov exponent has large fluctuation about zero, the system is nonhyperbolic. The unshadowability of the short periodic SPT is obvious. A shadowable chaotic pseudotrajectory must be similar to the true chaotic trajectory, e.g. being sensitive to noise and possessing ergodicity rather than a single periodic cycle with short period. In the simulation, in order to get an exactly periodic force, 200 values of the force $A\sin(2\pi ft)$ within a period are stored in a vector and used repeatedly to drive the logistic map. If we use $A\sin(2\pi ft)$ directly in the computer simulation, the obtained time series of driving force cannot be exactly periodic. Thus, according to Eq. (13), an interesting results is that the SPT obtained cannot be periodic.

## 4. Conclusion

In this Letter, the numerically truncated trajectory for a periodically driven chaotic system is discussed. With a low-frequency driving force, the dynamics of the system repeatedly alternate between expansion and contraction for long stretches of time. Compared to autonomous systems, the truncated trajectory of such a system shows some different properties. The deeply converging dynamics within a driving period can contract various trajectories to an single pseudotrajectory. An SPT is more easily observed numerically with a lower frequency driving force. Driven by an exactly periodic force, the SPT is numerically periodic. Its period is short and independent of truncation error in some scales. A short period SPT is obviously unshadowable. It cannot give us correct properties for the chaotic system. However, a noise-disturbed SPT can approach the real invariant density of the system and thus provide reasonable results. On the other hand, if the driving force is not exactly periodic, an SPT cannot collapse to a periodic cycle.

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