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Properties of phase locking with weak phase-coherent attractors

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Abstract

In coupled chaotic oscillators, the synchronization with weak phase-coherent attractors is different from that with strong phase-coherent ones. The properties of phase locking for weak phase-coherent attractors are studied with examples. For a small parameter mismatch, transition to phase locking is close to the position where the second zero Lyapunov exponent becomes negative and one of the positive Lyapunov exponents becomes zero simultaneously. However, for a large mismatch, it occurs at a farther position evidently. These results lead to a better understanding of the properties of synchronization in coupled oscillators. © 2001 Published by Elsevier Science B.V.

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1. Introduction

Synchronization of interacting oscillators is of fundamental interest for many applications of nonlinear dynamics in chemistry [1], electronics [2] and biology [3]. A complete coincidence of states in the systems of concern is referred as *complete or full synchronization* [4] while a time-shifted coincidence is called *lag synchronization* [5]. There is also *phase synchronization* (PS) [5–12], a phenomenon observed in a system formed by two coupled chaotic oscillators or in a single self-sustained chaotic oscillator subject to external force. This phenomenon usually refers to the situation that phases of the interacting systems appear to have a certain relation but the amplitudes remain chaotic and uncorrelated [11]. PS is an area of recent interest as it can help in understanding the synchronization of neuronal activity of remote areas in human brain [13] and multichannel nonlinear digital communications [14].

At first, PS behavior is discussed in weakly coupled chaotic oscillators that typically show weak correlation between their amplitudes. If phase coincidence is obtained at strong coupling, their amplitudes have a relatively large correlation. In some papers, the phase coincidence at strong coupling is also referred to as PS [15]. For better understanding, the term “*phase locking*” (PL) is used in this Letter to describe phase coincidence regardless of the correlation of amplitudes.

As PL of coupled oscillators leads to various applications, it is important to find out its properties for various *phase-coherent attractors* (PCA) [11,16]. Recent investigations mainly focus on the properties of PL for strong PCA [5,15] and so the relationship between the transition to PL and the corresponding Lyapunov ex-

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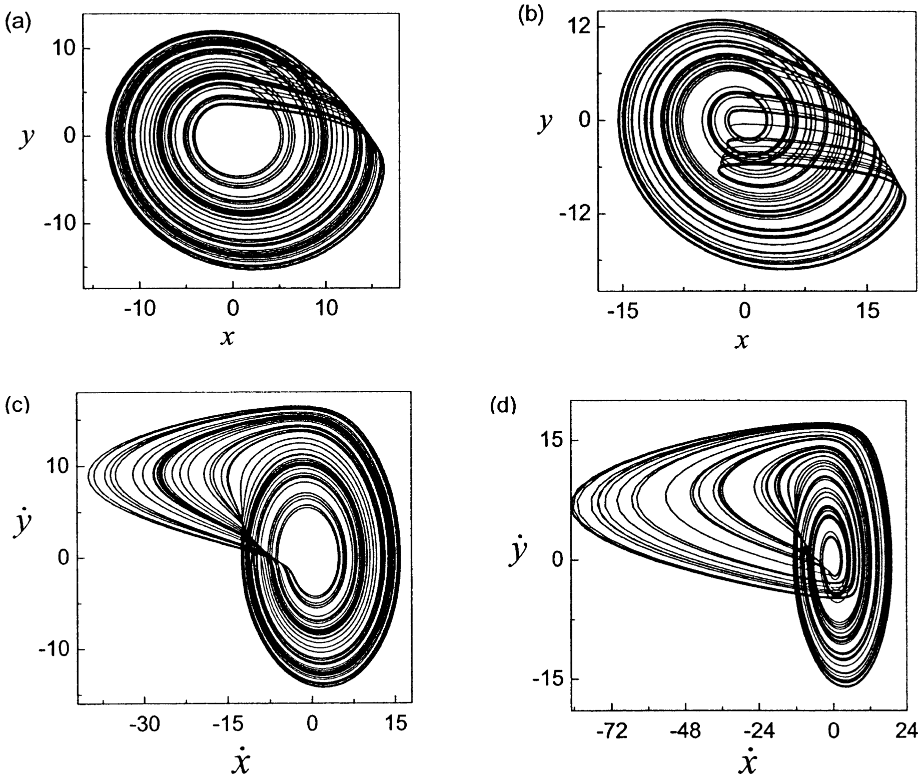

Fig. 1. Projections of phase portraits of the single Rössler oscillator. Strong PCA, $\alpha = 0.16$ in (a) (x, y)-plane and (c) $(\dot{x}, \dot{y})$-plane. Funnel attractor which belongs to weak PCA, $\alpha = 0.25$ in (b) (x, y)-plane and (d) $(\dot{x}, \dot{y})$-plane.

ponents are not clear for weak PCA. Strong PCA imply that the phase dynamics is relatively regular. In the power spectrum of one of the two variables that compose this type of attractors, there is a sharp peak on top of a broadband component. While for weak PCA, no sharp peak can be found in the spectrum [11,16]. The strength of phase coherence could be quantified by the diffusion constant defined in Eq. (6). In order to compare the properties of PL between strong and weak PCA extensively, a system formed by two coupled Rössler oscillators is used. This is because the attractors can vary from strong to weak PCA by adjusting a single parameter only. The weak PCA of Rössler oscillator always appears as a funnel shape, as shown in Fig. 1(b) [11,16–18]. Its trajectory sometimes makes a roundtrip around the origin in the (x, y)-plane but sometimes only a half of it is made. Because of the irregular trajectory, it is named funnel attractor and no effective method has been proposed to measure the instantaneous phase. In this Letter, we propose a method to measure the phase of attractors whose trajectories have a single direction of rotation, no matter whether they are phase-coherent or funnel. With this method, the properties of PL with various PCA can be investigated conveniently. We find that for a small parameter mismatch, the synchronization properties of weak PCA are different from those of strong ones. For strong PCA, the transition to PL is close to the position where the first zero Lyapunov exponent goes negative [15]. However, for weak PCA, it is near to the position where the second zero Lyapunov exponent turns negative and one of the positive Lyapunov exponents becomes zero simultaneously. We also show that for a large parameter mismatch and weak PCA, the transition to PL occurs at a farther position evidently. Furthermore, we take two coupled Lorenz attractors as another example to emphasize that the properties of PL can be found in a variety of oscillators with weak PCA.

Finally, some conclusions are obtained. These results can lead to a better understanding of the properties of PL for various PCA.

2. Phase locking between two coupled Rössler oscillators

2.1. Phase measurement for trajectories with a single direction of rotation

A single Rössler oscillator [19] is considered first:

$$\begin{aligned}\dot{x} &= -\omega y - z,\\ \dot{y} &= \omega x + \alpha y,\\ \dot{z} &= 0.1 + z(x - 8.5),\end{aligned} \tag{1}$$

where the natural frequency $\omega = 1$. The coherence of the attractor is determined by the value of the parameter α. When $\alpha = 0.16$, the attractor appears to have a strong phase coherence, as shown in Fig. 1(a). However, if $\alpha = 0.25$, a funnel attractor with weak phase coherence is observed and is shown in Fig. 1(b). The trajectory of the funnel attractor is found to encircle the origin of the (x, y)-plane completely at some particular moments, but sometimes it performs this only partially. These irregular phase shifts can be interpreted as the interruption of phase coherence by a large fluctuation [16].

It is difficult to obtain the instantaneous phase with existing phase definitions [11]. Therefore we have to develop a novel method to measure this value before investigating the properties of PL at weak PCA. The phase measurement for funnel attractors could be developed from a similar definition for attractors with strong phase coherence. Suppose that an attractor in the (x, y)-plane has a center of rotation (x_0, y_0). Then its phase can be defined as

$$\phi_T = \arctan\left(\frac{x - x_0}{y - y_0}\right). \tag{2}$$

The advantage of this definition is that the phase at time t is only determined by the state variables x and y at that moment and so the whole time series is not required. However, this definition is still a function of the center of rotation. If the phase ϕ_T of a chaotic flow with multiple centers of rotation needs to be calculated, the center point (x_0, y_0) in Eq. (2) should be changed accordingly. This requires the exact knowledge of the centers of rotation along the trajectory.

The basic idea of the proposed definition is that the difference between the nearby points $x(t) - x_0$ and $x(t + \delta t) - x_0$, rather than the single point $x(t) - x_0$, is taken into consideration. Here δt is an infinitely small time interval. The purpose of this change is to remove the center of rotation from Eq. (2). In the (x, y)-plane, the phase is then measured as

$$\begin{aligned}\phi_D &= \lim_{\delta t \to 0} \arctan\left(\frac{x(t + \delta t) - x(t)}{y(t + \delta t) - y(t)}\right)\\ &= \arctan\left(\frac{\dot{x}(t)}{\dot{y}(t)}\right).\end{aligned} \tag{3}$$

Obviously, for a constant-speed rotation with amplitude A and angular velocity ω, $x(t) = A\cos(\omega t)$ and $y(t) = A\sin(\omega t)$. As a result, the phases under the two different definitions are $\phi_T = \omega t$ and $\phi_D = \omega t + \pi/2$, respectively. This shows that the novel definition is consistent with the phase found by Eq. (2).

The phase value measured using Eq. (3) does not require the knowledge of the rotation center, but the curvature of the trajectory with a single direction of rotation. Comparing Eqs. (2) and (3), it is evident that if the funnel attractor is analyzed in the $(\dot{x}, \dot{y})$-plane instead of the (x, y)-plane, the new plane should contain an attractor with a center of rotation at $(0, 0)$. This is plotted in Fig. 1(d). On the other hand, we can also use Eq. (3) to measure the instantaneous phase of the attractor with strong phase coherence. The attractor in Fig. 1(a) is re-drawn in the $(\dot{x}, \dot{y})$-plane, as shown in Fig. 1(c). In this plane, both the strong PCA and the funnel attractors have the origin as the center of rotation. The only difference is the variation of amplitudes. In Fig. 1(c), the relatively slow variation of amplitude corresponds to strong PCA. However, Fig. 1(d) shows that the variation of amplitude is much faster in the funnel attractor. Evidently, the measurement given by Eq. (3) can be used for trajectories with a single direction of rotation, no matter whether the attractor is strong phase-coherent or funnel. However, this measurement has its limits. For trajectories with multiple directions of rotation like Lorenz attractor [20], a single rotation center cannot be found in the $(\dot{x}, \dot{y})$-plane. In specific conditions such as oscillation quenching caused by very large coupling strength [21], the attractor converges to one point and evidently Eq. (3) could

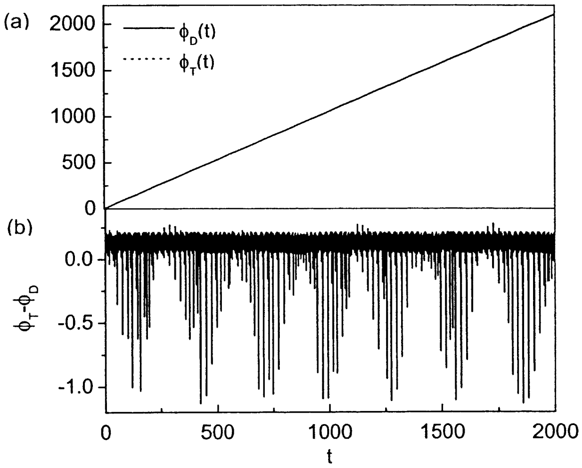

Fig. 2. The time evolution of the phase and phase difference for the strong phase-coherent Rössler oscillator. (a) The solid and the dotted lines represent the phases ϕ_D and ϕ_T, respectively. They have the same initial value. (b) The difference of the two phases in (a) versus time.

not be used. In this Letter, all coupling strength used in simulations is far from the transition for oscillation quenching to occur. Therefore, the measurement is always effective in our numerical simulations.

As both phase measurements (Eqs. (2) and (3)) can give the phase value for strong PCA, the phase of the trajectory shown in Fig. 1(a) is calculated using these two equations respectively. The results are plotted in Fig. 2(a), with both phases having the same initial value. In this figure, the dotted and the solid lines represent the simulation results using Eqs. (2) and (3), respectively. Their values are so close to each other that virtually only one straight line appears in the figure. In order to view their little difference clearly, the value of $\phi_T - \phi_D$ is plotted in Fig. 2(b). The difference keeps a small value within the duration of simulation. Moreover, the fluctuation has similar small periods at a large time scale. This implies that although the two phases ϕ_T and ϕ_D do not coincide microscopically, they have equal average growth rates.

2.2. Properties of phase locking with various diffusion constant

We now consider two coupled *Rössler* oscillators [6,19]:

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),$$
$$\dot{y}_{1,2} = \omega_{1,2}x_{1,2} + \alpha y_{1,2},$$
$$\dot{z}_{1,2} = 0.1 + z_{1,2}(x_{1,2} - 8.5), \tag{4}$$

where the natural frequencies $\omega_{1,2} = \omega_0 \pm \Delta\omega$ and $\omega_0 = 1$. In particular, $\Delta\omega$ is the frequency mismatch and ε is the coupling strength of the interaction between the two oscillators.

The phenomenon of PL between coupled oscillators is considered as $|n_1\phi_1 - n_2\phi_2| <$ const, where n_1, n_2 are integers and ϕ_1, ϕ_2 are phases of the two oscillators. In this Letter, our study is restricted to the case $n_1 = n_2 = 1$ with $\phi_{1,2}$ obtained by Eq. (3).

PL with strong PCA has been investigated extensively [20–23]. However, it differs substantially from that with weak PCA. We take $\alpha = 0.25$ as an example to simulate some of its statistical properties and plot the results in Fig. 3. In our simulation, the frequency mismatch is always set to $\Delta\omega = 0.02$ unless otherwise stated. The mean (observed) frequency is given by [21]

$$\Omega_{1,2} = \langle\dot{\phi}_{1,2}\rangle = \lim_{T\to\infty} \frac{\phi_{1,2}(T) - \phi_{1,2}(0)}{T}. \tag{5}$$

In Fig. 3(a), $\Delta\Omega = \Omega_2 - \Omega_1$ and simulations are performed with $T = 1 \times 10^5$ after neglecting the initial 1×10^4 time. If not specified, this condition is used throughout this Letter. A further increase of coupling strength makes $\Delta\Omega \to 0$ eventually and PL is thus obtained. At the transition, the coupling strength $\varepsilon = \varepsilon^c = 0.172$, as found from Fig. 3(a). Note that the transition to PL is always smeared and observed only as a tendency, or as a temporary event on some finite time intervals because the average laminar length increases exponentially (here, the laminar length is the time elapsed between two successive $\pm 2\pi$ jumps) [24]. The four largest Lyapunov exponents are shown in Fig. 3(b). The transition to PL (ε^c) is close to the position where the second zero Lyapunov exponent becomes negative and one of the two positive Lyapunov exponents reduces to zero simultaneously. This is denoted as ε_3^λ. The maximum absolute difference $|\Delta x(t)|_{\max}$ in the x value of the coupled oscillators is plotted in Fig. 3(c). When the coupling strength increases from zero, the value of $|\Delta x(t)|_{\max}$ decreases. This shows that their trajectories are highly correlated at the transition to PL.

In order to further uncover the relationship between the transition to PL and the strength of phase coherence, we introduce the diffusion constant $D_{1,2}$ that is

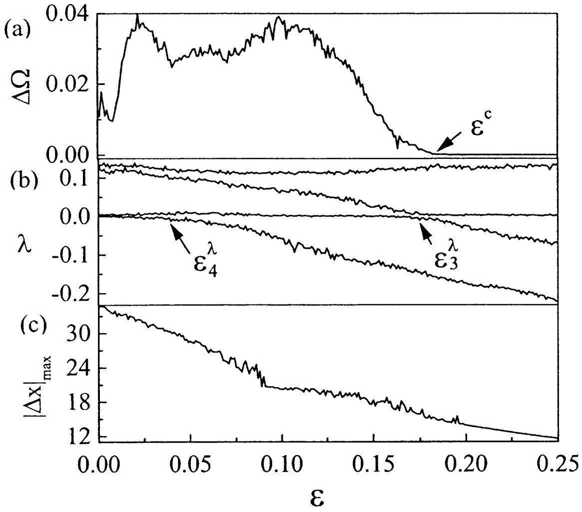

Fig. 3. Some statistical properties of the two coupled Rössler oscillators versus the coupling strength ε with $\alpha = 0.25$, $\omega_1 = 0.98$ and $\omega_2 = 1.02$. (a) The frequency difference $\Delta\Omega$ between two coupled oscillators. (b) The four largest Lyapunov exponents of the coupled oscillators. (c) The maximum absolute difference $|\Delta x|_{\max}$ of two trajectories with funnel attractors. Here, ε^c represents the coupling strength at transition to PL; ε_4^λ represents the coupling strength at the transition where the first zero Lyapunov exponent becomes negative; ε_3^λ represents the coupling strength at the transition where the second zero Lyapunov exponent becomes negative and one of the positive Lyapunov exponents reduces to zero simultaneously.

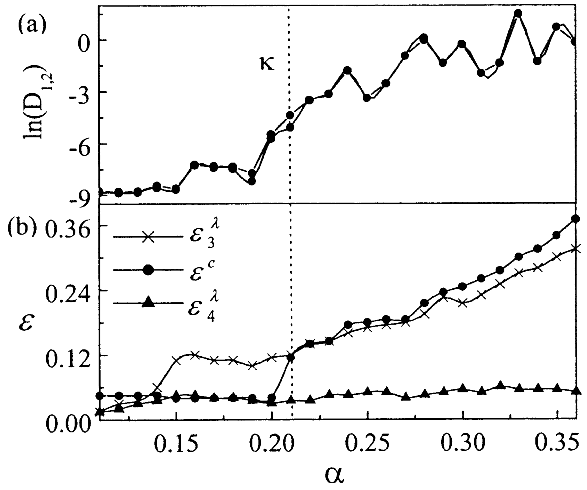

Fig. 4. Some statistical properties of the attractors versus the parameter α. Here, $\omega_1 = 0.98$ and $\omega_2 = 1.02$. (a) The logarithm of the diffusion constant $\ln(D_{1,2})$. It is simulated with zero coupling strength, i.e., $\varepsilon = 0$. (b) The value of coupling strength at various transitions. The vertical dotted line κ indicates the transition from strong to weak PCA.

always used to measure the strength of diffusion [11]. At large t, it is defined by

$$\left\langle \left(\phi_{1,2}(t) - \phi_{1,2}(0) - \Omega_{1,2}t\right)^2 \right\rangle_{\phi(0)} \sim D_{1,2}t. \tag{6}$$

Here, $\langle\cdot\rangle_{\phi(0)}$ denotes the average value that the simulations are taken at different initial points $\phi_{1,2}(0)$ but with constant time duration t. In this Letter, we always set $t = 5 \times 10^3$ in the simulations. We calculate $D_{1,2}$ of the two oscillators with $\varepsilon = 0$ and plot the results in Fig. 4(a). The two curves are close to each other for they only have a small mismatch of natural frequencies. Fig. 4(b) shows three curves versus the parameter α from 0.11 to 0.36, i.e., from strong to weak PCA. From the curves of $\ln(D_{1,2})$, it is evident that there are two main regions of diffusion constant. One is $\ln(D_{1,2}) \sim -7.5$ while the other is $\ln(D_{1,2}) \sim -1.5$. Thus the region to the left of the vertical dotted line κ corresponds to strong PCA while the right part corresponds to weak attractors. In the left part, ε^c is close to ε_4^λ when $0.15 \leqslant \alpha \leqslant 0.20$. The position of ε_4^λ has been located in Fig. 3(b). This is the same as the results presented in other papers discussing strong PCA [5]. When $\alpha < 0.15$, the two interactive attractors are in period when ε increases slightly from zero. Hence, all Lyapunov exponents are negative. The transition to PL occurs evidently for parameters larger than ε_4^λ. When $\alpha > 2.0$, i.e., in the right region of the dotted line κ, $\ln(D_{1,2})$ increases rapidly. The strong PCA then becomes weak. The corresponding transition to PL also jumps up to values close to ε_3^λ. The value of ε^c as well as ε_3^λ increases with $\ln(D_{1,2})$. This implies that for stronger diffusion constant a larger coupling for the transition to PL is required. When $\alpha > 0.15$, ε_4^λ maintains at a nearly constant value (about 0.045) in both strong and weak PCA.

The above-mentioned phenomena are explained briefly as follows: we can rewrite Eq. (4) in terms of variables $A_{1,2}, \phi_{1,2}, z_{1,2}$, where $A_{1,2}$ are amplitudes [5]. For strong PCA, $A_{1,2}$ varies so slowly that it can be considered as constant when compared with the fast variation of the phase $\phi_{1,2}$. Thus the transition to PL can be obtained as $\varepsilon^c \approx 2\Delta\omega$ [5]. The simulation results plotted in Fig. 4(b) show that $\varepsilon^c \approx 0.045$ at $\alpha \leqslant 0.20$ and the difference of natural frequencies is $\Delta\omega = 0.02$. However, for weak PCA, the amplitude

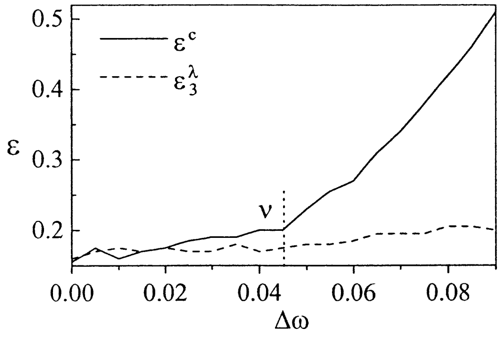

Fig. 5. The relationship between the mismatch of natural frequencies $\Delta\omega$ vs. coupling strength at transition to PL (ε^c) and the transition that the second zero Lyapunov exponent goes negative (ε_3^λ). The left to the vertical dotted line ν can be considered as one of small parameter mismatch while the right part has large parameter mismatch.

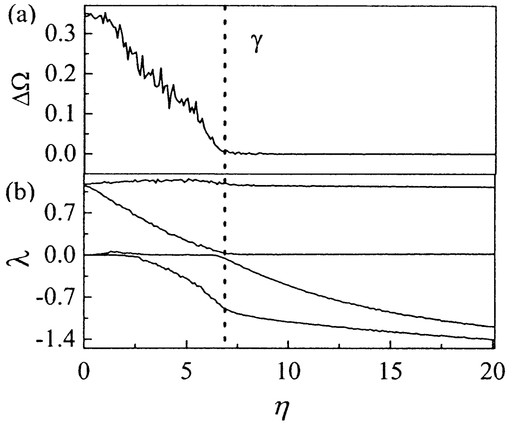

Fig. 6. (a) The frequency difference $\Delta\Omega$. (b) The four largest Lyapunov exponents λ of two coupled Lorenz oscillators vs. the coupling strength η in $(u-z)$-plane. The vertical dotted line γ indicates the transition to PL.

varies far more quickly than for strong attractors, as observed in Figs. 1(a) and (b). It can no longer be considered as constant and so the relationship between ε^c and $\Delta\omega$ is different from that for strong PCA. In the following, we will investigate these relationships by simulations and analysis.

Taking $\alpha = 0.25$ as an example, the frequency mismatch $\Delta\omega$ varies from 0.0 to 0.09 as shown in Fig. 5, with ε^c and ε_3^λ plotted as solid and dashed curves, respectively. For the region of small mismatch at the left of the dotted line ν, the values of ε^c and ε_3^λ are close to each other. While for the region of large mismatch at the right of the dotted line, ε^c increases almost linearly with a large slope but ε_3^λ increases slowly.

From Fig. 5, it is evident that with random initial conditions and $\Delta\omega = 0$, two identical Rössler oscillators could not achieve PL until $\varepsilon = 0.155$. However, for strong PCA, a small increase of the coupling strength from zero is sufficient to make the coupled attractors reach PL. These phenomena are mainly caused by the different values of their diffusion constant shown in Fig. 4(a).

3. Phase locking between two coupled Lorenz oscillators

The properties of PL with weak PCA can also be found in a variety of oscillators. Here, two coupled Lorenz oscillators are chosen as another example that shows the properties of PL. The system equations are

$$\begin{aligned}
\dot{x}_{1,2} &= 10.0(y_{1,2} - x_{1,2}) + \eta(x_{2,1} - x_{1,2}),\\
\dot{y}_{1,2} &= \omega_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2},\\
\dot{z}_{1,2} &= -3.0z_{1,2} + x_{1,2}y_{1,2},
\end{aligned} \tag{7}$$

with $\omega_{1,2} = 37 \pm \Delta\omega$ and $\Delta\omega = 1$. Here, η is the coupling strength. In the (u, z)-plane with $u = (x^2 + y^2)^{1/2}$ [20], the phase can be obtained by Eq. (3). With $\eta = 0$, $D_{1,2}$ of the two Lorenz oscillators are 0.09 and 0.10, respectively. Evidently, they correspond to weak PCA. Fig. 6(b) shows the four largest Lyapunov exponents λ of the coupled Lorenz oscillators versus the coupling strength η. The results of Figs. 6(a) and (b) also show that the transition to PL is closed ε_3^λ.

4. Conclusions

The properties of PL for weak PCA have been studied with mutually coupled oscillators. An efficient method has been developed to measure the instantaneous phase of attractors with trajectories in a single direction of rotation. Furthermore, we investigate two coupled Rössler oscillators with a small parameter mismatch and find that for strong PCA, the transition to PL is always close to the point where the first zero Lyapunov exponent turns negative. While for weak PCA, the transition is always near to the position where the second zero Lyapunov exponent becomes negative and one of the positive Lyapunov exponents becomes zero simultaneously. However, for a large parameter mismatch and weak PCA, the transition to PL is much farther away from this position. At last, we take two coupled Lorenz oscillators as another example to show that the properties of PL for weak PCA can also be found in a variety of coupled oscillators.

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