# Phase synchronization in discrete chaotic systems

J. Y. Chen,[1,2] K. W. Wong,[1] Z. X. Chen,[2] S. C. Xu,[2] and J. W. Shuai[3]
[1]Department of Electronic Engineering, City University of Hong Kong, Kowloon Tong, Hong Kong
[2]Department of Physics, Xiamen University, Xiamen, China
[3]Department of Biomedical Engineering, Case Western Reserve University, Cleveland, Ohio 44106

(Received 3 August 1999; revised manuscript received 1 December 1999)

A simple and instantaneous phase definition is proposed for the study of discrete maps by taking the change of chaotic signal at each iteration time as a vector. With such a definition, an exact phase can be calculated at any iteration time for any scalar signal or two-dimensional vector of interest. As examples, the phase synchronization behavior is discussed for a two-dimensional globally coupled map lattice and a one-way coupled map lattice.

PACS number(s): 05.45.−a, 47.54.+r

Recently, the concept of phase, as well as phase synchronization (PS), has been generalized to the study of chaotic systems [1,2]. In particular, PS has been studied in nonlinear neural [3,4], cardiac [5], and ecological systems [6,7]. It is also observed in oscillations between respiratory and cardiac rhythms [5] or between brain activity and the signals from the flexor muscle [3]. The subthreshold chaotic oscillation of electrically coupled inferior olivary neurons *in vitro* has also been examined from the view of PS [4]. The results of these studies suggest that PS plays an important role in the behavior of chaotic systems.

The PS of autonomous continuous-time systems is usually defined as the appearance of a certain relation between the phases of the interacting systems while the amplitudes remain chaotic and are, in general, noncorrelated. As many systems under study are discrete-time systems, it is important if the concept of phase can be extended to discrete-time maps so that their PS behavior can be investigated. In Ref. [8], PS has been discussed for discrete maps by checking if the system scalar signals simultaneously show local maxima or minima. With this process, the phase is only defined at the local extremes of scalar signals and does not apply to multidimensional vectors.

In this paper, a simple and instantaneous phase definition is proposed for the study of discrete maps. With such a definition, an exact phase can be calculated at any iteration time for any scalar signal or two-dimensional vector of interest. As a result, PS can be discussed in a more flexible and quantitative manner.

We first consider a discrete map $\vec{x}(t+1)=\vec{F}(\vec{x}(t))$ where $\vec{x}(t)=[x_1(t),x_2(t),\ldots,x_N(t)]$ is the state vector and $N$ is the dimension of the map. The phase of a two-dimensional vector in the $s_1-s_2$ plane is defined as

$$\phi(t+1)=\arctan\left(\frac{s_2(t+1)-s_2(t)}{s_1(t+1)-s_1(t)}\right), \tag{1}$$

$$\psi(t)=\phi(t)+2\pi m(t); \tag{2}$$

$i$ is the lattice site, $t$ is the iteration time, and $\Psi(t)$ is the phase value. The signals $s_{1,2}$ can be any linear combination of the elements in $\vec{x}(t)$, i.e., $s_{1,2}=\Sigma_{i=1}^{N}a_i^{1,2}x_i$ with $a_i^{1,2}\in R$. If we want to discuss the phase of a scalar signal, e.g., $s_2(t)$, we can simply let $s_1(t)=t$. The definition of the phase variable $\phi(t)$ in Eq. (1) indicates that the chaotic signal at time $t$ is taken as the center of that at time $t+1$. In order to make the phase continue to increase in a specific direction, the integer $m(t)$ is chosen as

$$m(t+1)=\begin{cases} m(t)+1 & \text{if } \phi(t+1)<\phi(t) \\ m(t) & \text{otherwise} \end{cases}, \quad t=1,2,3,... \tag{3}$$

with $m(1)=0$.

The mechanism caused by this definition is shown in Figs. 1(a) and (b). In Fig. 1(a), $s(t)$ is a vector on the $s_1-s_2$ plane while $r$ is a measure of the length of this vector. The phase is independent of $r$. This is further illustrated in Fig. 1(b) in which the starting points of all the vectors are at the origin. Thus our phase definition only concerns the angle between the current vector and the $s_1$ axis and does not depend on the length of the vector. With this process, the PS of discrete maps is defined as the appearance of a certain relation between the phases of the coupled maps while the distances between them remain chaotic.

In the rest of this paper, the PS phenomenon in two typical systems is investigated. In the first example, we reconsider the PS states of the global coupled map lattices studied in Ref. [8], where the signal of interest is a scalar. Simulation results show that our definition of phase is consistent with that in Ref. [8]. In the second example, we examine the PS in two one-way coupled map lattices, where the signal of interest is a two-dimensional vector.

*Example 1.* We first consider an ensemble of $N$ coupled one-dimensional map lattices, each formed by $L$ logistic maps [8]. In this system, the state $x_k^i$ of the $k$th map ($k=1,...,L$) in the $i$th lattice ($i=1,...,N$) evolves through iterations according to the following formula:

$$\begin{aligned} x_k^i(t+1)=&(1-2\varepsilon_1-2\varepsilon_2)f_k^i(x_k^i(t))+\varepsilon_1 f_k^i(x_{k-1}^i(t))\\ &+\varepsilon_1 f_k^i(x_{k+1}^i(t))+\varepsilon_2 f_k^i(M^{i-1}(t))\\ &+\varepsilon_2 f_k^i(M^{i+1}(t)). \end{aligned} \tag{4}$$

 

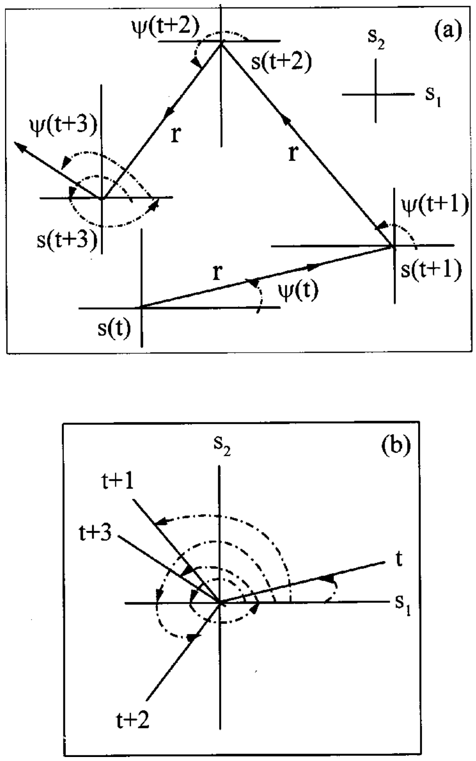


FIG. 1. The mechanism caused by the proposed phase definition. (a) $\psi(t)$, $\psi(t+1)$, $\psi(t+2)$, and $\psi(t+3)$ are the phase states of the points $s(t)$, $s(t+1)$, $s(t+2)$, and $s(t+3)$ on the $s_1-s_2$ plane while $r$ is the distance between two points. (b) All the phase states are considered at a specific direction (counterclockwise) with respect to the origin.

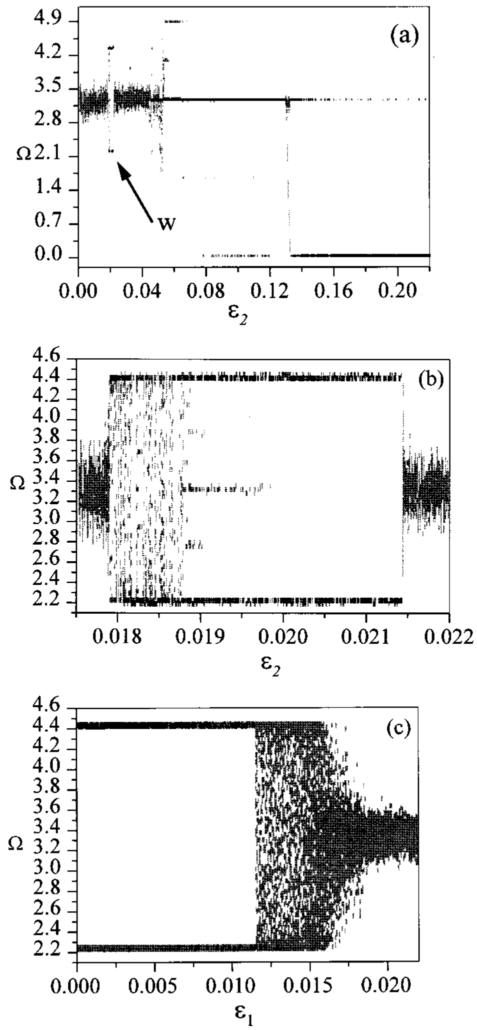


FIG. 2. Phase phenomenon among the global signals. (a) $\varepsilon_2$ versus the average frequencies $\Omega$, with $\varepsilon_1=0$. Region $w$ corresponds to the two clusters of the PS phenomenon. The step size is $\Delta\varepsilon_2=0.5\times10^{-3}$. (b) An enlargement of region $w$ with $\Delta\varepsilon_2=0.5\times10^{-4}$. (c) $\varepsilon_1$ versus $\Omega$ with $\varepsilon_2=0.021$ and $\Delta\varepsilon_1=0.5\times10^{-4}$.

In Eq. (4), $\varepsilon_1$ and $\varepsilon_2$ are the coupling parameters, $f_k^i$ is the logistic map defined by $f_k^i(x)=\mu_k^i x(1-x)$ with $0<\mu_k^i\leqslant 4$ [8], and $M^i(t)=(1/L)\sum_{k=1}^{L} x_k^i(t)$ is the mean field of the $i$th lattice at time $t$.

If $\varepsilon_1=\varepsilon_2=0$, Eq. (4) describes the dynamics of $N\times L$ independent logistic maps. However, if $\varepsilon_1\neq 0$ and $\varepsilon_2=0$, this system can be considered as a collection of $N$ independent one-dimensional lattices of logistic maps. The maps are coupled within each lattice by means of a diffusive term. Furthermore, if $\varepsilon_2\neq 0$, each lattice is coupled to its nearest two lattices with the mean field. For all of the elements within a lattice, the values of such couplings are the same, e.g., $\Gamma^i(t)=\varepsilon_2[f_k^i(M^{i-1}(t))+f_k^i(M^{i+1}(t))]$.

Due to the coupling of $\Gamma^i$, the collective PS can occur between the mean fields $M^i(t)$ and $M^j(t)$ $(i\neq j)$ while all the maps $x_k^i(t)$ and $x_k^j(t)$ $(k=1,\ldots,L)$ are uncorrelated to each other [8]. It has also been observed that there are two PS clusters of lattices. In each PS cluster, the mean fields of lattices simultaneously show local maxima or minima. With our phase definition, this phenomenon can be clearly observed. Furthermore, the phase states can be calculated instantaneously at any discrete time. This characteristic of our phase definition allows for a more detailed analysis of the system.

All the results presented here correspond to the condition that $\mu=4$, $N=L=100$. The phase state $\psi_i(t)$ of the scalar mean field $M^i(t)$ is calculated using Eqs. (1)–(3) with $s_1(t)=t$ representing the iterating time and $s_2(t)=M^i(t)$. The average frequency is defined as $\Omega_i=\psi_i(t)/t$ with $t\to\infty$ [1]. In computer simulations, $t=5000$ is large enough to obtain a precise value of $\Omega$. Therefore $t$ is always set to 5000 in the calculation of $\Omega$ unless otherwise specified. First we let $\varepsilon_1=0$ and $\varepsilon_2$ is allowed to increase from 0 to 0.22 in a fixed step size. A plot of $\Omega_i$ versus $\varepsilon_2$ is given in Fig. 2(a). If two or more signals have the same value of $\Omega$, they are considered to be in PS state. The iterations start from random initial conditions. The initial 1000 iterations are omitted and the phases of $M$ in the subsequent 5000 iterations are calculated. If more than 40 signals of $M$ have the same value of $\Omega$, the points are shown in black. Otherwise, they are plotted in gray. The phase begins in an unsynchronized state. When $\varepsilon_2$ is in the neighborhood of 0.02, there is a narrow region $w$ in which all the mean fields $M$ show two clusters. It indicates that there are two clusters of PS states. This phenomenon will be discussed in detail in the following paragraph. A further increase in $\varepsilon_2$ causes all the mean fields to be desynchronized. When $\varepsilon_2>0.04$, most of the fields $M$ begin to converge to a cluster of synchronization, except some lattices indicated by the gray points. When $\varepsilon_2$ exceeds 0.13, the average frequencies of most signals approach zero. From this figure, we find that the PS in discrete maps is different from that in autonomous systems: in discrete map, alternating intervals of PS or non-PS phenomenon can be observed as the coupled strength is increased. However, in the autonomous system, there exists a critical threshold of the coupling strength to switch incrementally from non-PS to PS to full synchronization [10].

The region $w$ in Fig. 2(a) is a region of weak synchronization [8]. An enlargement of this region can be found in Fig. 2(b). With the increase of $\varepsilon_2$, the system evolves from the unsynchronized state to two clusters of PS states and then back to the unsynchronized state. Our results are consistent with those reported in Ref. [8] but provide a clearer overview of the characteristics of PS.

We further investigate the two clusters of PS states by raising the value of $\varepsilon_1$. Let $\varepsilon_2=0.021$ and $\varepsilon_1$ is increased from 0 to 0.022. The simulation results are shown in Fig. 2(c). In this figure, it is clearly shown that the two-cluster PS phenomenon is destroyed by the increase of $\varepsilon_1$. This implies that an increase in coupled strength can destroy the PS phenomenon among mean fields.

*Example 2.* The one-way coupled map lattices have been investigated extensively [9]. The PS behavior between two two-dimensional vectors is discussed here. The map lattices are described by

$$x_i(t+1)=(1-\varepsilon_1)f_1(x_i(t)) + \varepsilon_1 f_1(x_{i+1}(t)) + c(x_1(t)-y_1(t)), \tag{5}$$

$$y_i(t+1)=(1-\varepsilon_2)f_2(x_i(t)) + \varepsilon_2 f_2(x_{i+1}(t)) + c(y_1(t)-x_1(t)), \tag{6}$$

where $i\in(1,2,...,N)$ and $c$ represents the strength of coupling between the two nonidentical systems. The parameters $\varepsilon_1$ and $\varepsilon_2$ represent the strength of coupling among lattices in the two systems, respectively. Here we choose $N=10$, $f_1(x)=\alpha_1 x(1-x)$, and $f_2(x)=\alpha_2 x(1-x)$ with $\alpha_1=3.7$, $\alpha_2=3.8$, and $\varepsilon_1=\varepsilon_2=0.6$.

In this example, the phase $\psi_x(t)$ is defined for the two-dimensional vectors on the $x_1-x_2$ plane, i.e., with $s_1=x_1$ and $s_2=x_2$ in Eq. (1); while $\psi_y(t)$ is on the $y_1-y_2$ plane. Note that similar results are obtained with different choices of $s_1$ and $s_2$. The initial condition of the two one-way coupled map lattices is randomly set. After the first 10 000 iterations are omitted, the phase difference $\theta(t)=\psi_x(t)-\psi_y(t)$ is calculated and plotted in the outer part of Fig. 3(a). When $c=0.01$, the system is in an unsynchronized state. But when $c=0.06$, the two coupled map lattices achieve PS. However, this synchronization cannot be maintained if $c$ continues to increase. For example, when $c=0.08$, the PS is destroyed and the phase difference increases to negative direction. In the inner part of Fig. 3(a), the trajectory of Eq. (5) on the $x_1-x_2$ plane with $c=0$ is plotted. Although the PS behavior is generalized from coupled chaotic oscillators to coupled maps, there is a difference between these two kinds of systems. In order to define the phase, the trajectory of each coupled chaotic oscillator is normally required to have only a single rotation center [1,2].

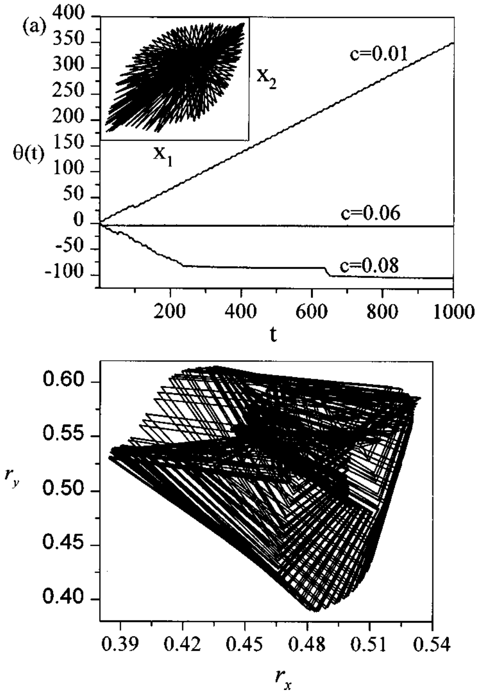


FIG. 3. Plots of simulation results with two one-way coupled map lattices. (a) The inner plot shows one of the attractors on the $x_1-x_2$ plane while the outer plot is phase difference $\theta(t)$ versus time $t$ at different values of coupled strength. (b) The trajectories on the $r_x-r_y$ plane with $c=0.06$.

However, as the trajectory of the discrete map is always formed by a large number of discrete dots without any rotation, there is no such restriction.

In order to investigate the characteristics of PS, a plot of the trajectories on the $r_x-r_y$ plane is shown in Fig. 3 (b) with $c=0.06$. Here $r_{x,y}$ are the distances between two iterating points, as shown in Fig. 1 (a). If the two coupled map lattices are in full synchronization, there will be a straight line corresponding to $r_x=r_y$ in the plot. However, this is not found and so the two systems are in PS state only.

The Lyapunov exponents are calculated and plotted for this example because they can reflect how the PS manifests itself in a chaotic system [1]. Figure 4(a) shows the two largest Lyapunov exponents $\lambda_1$ and $\lambda_2$ of the two coupled map lattices of Eqs. (5) and (6) against the coupling $c$. Figure

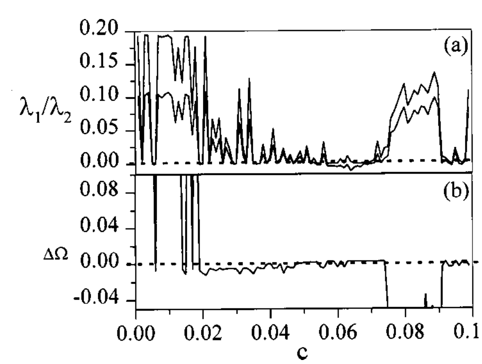


FIG. 4. Plots of simulation results with two coupled map lattices. (a) The two largest Lyapunov exponents $\lambda_1$ and $\lambda_2$ of the two coupled map lattices, and (b) the average frequency difference $\Delta\,\Omega$ vs the coupling $c$.

4(b) is a plot of the average frequency difference of the two phases versus the coupling $c$. Here the average frequency difference is defined as $\Delta\Omega=\Omega_1-\Omega_2=(\psi_1-\psi_2)/t$ with $t\to\infty$. Similar to the calculation of $\Omega$, $t=5000$ is large enough to obtain a precise value. When $\Delta\Omega\to 0$, it indicates the occurrence of PS between the two vectors. From the relations between Figs. 4(a) and (b), it can be found that PS always corresponds to two largest Lyapunov exponents asymptotically approaching zero. For example, when $c=0.06$, the system is in PS and the calculation shows that $\lambda_1=5.3\times 10^{-4}$ and $\lambda_2=-4.96\times 10^{-3}$. On the other hand, non-PS is indicated by the case when both of the two largest Lyapunov exponents are positive and large. Therefore the PS phenomenon in discrete maps is based on weak chaotic states. In strong chaotic states such as hyperchaotic states, the two interacting systems cannot achieve PS.

In conclusion, a definition of phase is given for any scalar signal or two-dimensional vector of the discrete map at any discrete time. With such a definition, we show that PS can emerge in the collective behavior of an ensemble of chaotic coupled map lattices due to the mean field interaction. One- or two-cluster PS phenomenon is observed clearly with weak coupled strength. The results are consistent with those expected from an existing phase definition based on the statistics extracted from the time series generated by the system iterations [8]. Furthermore, the investigation on the PS phenomenon of two interacting one-way coupled systems shows that PS is observed intermittently. From the analysis of Lyapunov exponents, we have found that the PS phenomenon not only corresponds to weak coupled strength of the two interacting systems, but also relates to the weak chaotic states of the systems.

The authors would like to thank William C. Stacey for a critical reading of the manuscript. This work was supported by the Strategic Research Grant (Grant No. 7000773) provided by the City University of Hong Kong.

[1] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[2] K. J. Lee, Y. Kwak, and T. K. Lim, Phys. Rev. Lett. **81**, 321 (1998).

[3] P. Tass, M. G. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, and H. J. Freund, Phys. Rev. Lett. **81**, 3291 (1998).

[4] V. Makarenko and R. Llinas, Proc. Natl. Acad. Sci. USA **95**, 15 747 (1998).

[5] C. Schader, M. G. Rosenblum, J. Kurths, and H. Abel, Nature (London) **392**, 239 (1998).

[6] D. J. D. Earn, P. Rohani, and B. T. Grenfell, Proc. R. Soc. London, Ser. B **265**, 7 (1998).

[7] B. Blasius, A. Huppert, and L. Stone, Nature (London) **399**, 354 (1999).

[8] F. S. de San Roman, S. Boccaletti, D. Maza, and H. Mancini, Phys. Rev. Lett. **81**, 3639 (1998).

[9] T. Bohr and O. B. Christensen, Phys. Rev. Lett. **63**, 2161 (1989).

[10] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).