Reply to "Comment on 'Simple approach to the creation of a strange nonchaotic attractor in any chaotic system'"

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We have recently proposed a simple method to create a strange nonchaotic attractor with any chaotic system [Phys. Rev. E **59**, 5338 (1999)]. Such a system is controlled to switch periodically between a chaotic and a quasiperiodic attractor, each with an appropriate time duration. A topological condition for this approach is pointed out in the preceding Comment by Neumann and Pikovsky [Phys. Rev. E **64**, 058201 (2001)]. We show that this condition is not necessary if the durations are sufficiently long. Our approach is a general method to construct a strange nonchaotic attractor in any chaotic system.

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In Ref. [1], we addressed the problem of whether a general method to construct a strange nonchaotic attractor (SNA) in any chaotic system can be found. For simplicity, consider a discrete chaotic system

$$\mathbf{x}(t+1)=\mathbf{F}[\mathbf{x}(t),C_1], \quad (1)$$

where C_1 is a suitable control parameter. (In this paper, we use the same symbols employed in Ref. [2]). We proposed that an SNA can be generated in the following system:

$$\mathbf{x}(t+1)=\mathbf{F}[\mathbf{x}(t),C(t)]+A\sin(2\pi\omega t), \quad (2)$$

where ω is irrational, A should be small enough to maintain a chaotic attractor for $C(t)=C_1$ and a quasiperiodic attractor for $C(t)=C_2$. Suppose the maximum nontrivial Lyapunov exponent for the chaotic and quasiperiodic attractor is positive λ_1 and negative λ_2, respectively. By switching the parameter $C(t)$ periodically with $C(t)=C_1$ for duration T_1 and $C(t)=C_2$ for duration T_2, we showed that the nonchaotic attractor obtained can be a SNA when T_1 and T_2 are sufficiently longer than the transient process [1]. Thus $C(t)$ is typically a low-frequency periodic wave. Notice that in our method only the parameters in Eq. (1) are fixed, while all other parameters (C_2, T_1, T_2, A, and ω) in Eq. (2) are adjustable in order to construct a SNA.

In the preceding Comment (Ref. [2]), the authors claim that (*Claim 1*) a special topological property of the dynamics is required for the creation of the SNA in arbitrary T_1. There should be an unstable torus coexisting with the stable torus for $C(t)=C_2$, and the range of such an unstable torus should overlap with the band of the chaotic attractor for $C(t)=C_1$. For a system lack of such a condition, a SNA is only constructed in a finite region near the border to chaos. A phase diagram corresponding to this situation is given in Fig. 2 of Ref. [2]. The authors at last claim that (*Claim 2*), in general, it is not possible to construct a SNA in any chaotic system by applying the method of Ref. [1].

Consider the example of *Case A* with $C_1^A=-0.01$ in Ref. [2] and let $T_1=10^5$ with any $T_2>3.11\times10^4(\approx-\lambda_1^A T_1/\lambda_2)$. In this case, even a very small perturbation that is of the order of $\exp(-\lambda_1^A T_1)\approx10^{-1539}$ can be enlarged to 1 by the T_1-duration expanding dynamics. This indicates that a long segment of fully developed chaotic trajectory can be obtained during each T_1 period. The width of the band at the end of the regular part of iterations is about $2A=0.002$. With long T_1-duration expanding dynamics, this band can be expanded and folded sufficiently, resulting in a strange attractor. As a comparison, consider the graph plotted in Fig. 4 of Ref. [2]. It shows that the probability of observing a positive time-8000 Lyapunov exponent is smaller than 3×10^{-8}. According to Eq. (5) of Ref. [2], the chance of observing the positive time-10^5 Lyapunov exponent (i.e., $k=250$) in this SNA is about $\exp(-210)\approx10^{-91}$, which can be treated as zero. While in the present SNA example the positive time-10^5 Lyapunov exponent can be observed periodically in each driving period. This discussion shows that a SNA can be created for any $T_2>3.11\times10^4$ and $T_1=10^5$ without any special requirement on the topological property of the system. Hence, all the nonchaotic attractors obtained using our approach are SNAs if T_1 (more strictly, $T_1\lambda_1$) is sufficiently large. Long enough T_1 directly causes a long duration of expanding dynamics for generating strange attractor. It does not require any special topological condition in the system.

However, for the case of small T_1 (e.g., the examples discussed in Fig. 1 of Ref. [1] or Fig. 2 of Ref. [2]), a special topological requirement should be satisfied in order to obtain SNAs in a large region, as pointed out in Ref. [2]. Without such a condition, SNAs can only occur in a finite region near the border to chaos. Although it then follows the general SNA theory [3], the finite region for SNA is still larger than that obtained with most of the other SNA methods (listed in Ref. 3 of Ref. [2]). As an example, one can compare Fig. 2 in Ref. [2] with Fig. 1 in Ref. [4]. Another unique property of this method is that a SNA can be easily obtained in any high dimensional chaotic system, which is a challenging problem for most of the other SNA methods. Because a low-frequency driving force of $C(t)$, rather than the widely used sine wave with golden-mean frequency, is applied in our method. Such a force can easily induce a finite-time Lyapunov exponent fluctuating substantially around zero, which is the key for the generation of a strange attractor.

In summary, our reply to the preceding Comment in Ref. [2] is as follows: their *Claim 1* is only applicable to the case of small T_1. For sufficiently long T_1, SNAs can be created without any special topological condition in the system. We disagree with their *Claim 2*. The conclusion that our approach described in Ref. [1] is general for any chaotic system is in the sense that, for any given chaotic system, at least a SNA can be constructed with a set of suitably selected parameters of C_2, T_1, T_2, A, and ω. Even with a small T_1, one can certainly create a SNA at least near the border to chaos in any system.

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[3] A. Pikovsky and U. Feudel, Chaos **5**, 253 (1995).
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